Problem 75: Random quadratics (✓✓✓)

The random variable B is normally distributed with mean zero and unit variance. Find the probability that the quadratic equation

$$X^2 + 2BX + 1 = 0$$

has real roots.

Given that the two roots X_1 and X_2 are real, find, giving your answers to three significant figures:

(i) the probability that both X_1 and X_2 are greater than $\frac{1}{5}$;

(ii) the expected value of $|X_1 + X_2|$.

1988 Paper II

Comments

It is quite difficult to find statistics questions at this level that are not too difficult and are also not simple applications of standard methods. For example, χ^2 tests are not really suitable, because the theory is sophisticated while the applications are usually rather straightforward. Most questions in the probability/statistics area tend therefore to concentrate on probability, and many of these have a bit of pure mathematics thrown in.

Here, the random variable is the coefficient of a quadratic equation, which is rather pleasing. But you have to handle the inequalities carefully. The difficulty is increased by the conditional element: for parts (i) and (ii) you are only interested in the case of real roots.

If you don't have statistical tables handy, don't bother to find some: just leave the answers in terms of the probability, $\Phi(z)$, that a standard ($\mu = 0$, $\sigma = 1$) normally distributed random variable Z satisfies $Z \leqslant z$.

Solution to problem 75

The solution of the quadratic is

$$X = -B \pm \sqrt{B^2 - 1}$$

which has real roots if $|B| \geqslant 1$. Let $\Phi(z)$ be the probability that a standard normally distributed variable Z satisfies $Z \leqslant z$. Then the probability that $|B| \geqslant 1$ is (taking the two tails of the normal distribution)

$$\Phi(-1) + \big(1 - \Phi(1)\big) = 2 - 2\,\Phi(1) = 0.3174.$$

(i) We need the smaller root to be greater than $\frac{1}{5}$. The smaller root is $-B - \sqrt{B^2-1}$. Now provided $\sqrt{B^2-1}$ is real, we have

$$\begin{aligned}
-B - \sqrt{B^2-1} > \tfrac{1}{5} \quad &\Leftrightarrow \quad B + \tfrac{1}{5} < -\sqrt{B^2-1} \\
&\Leftrightarrow \quad (B+\tfrac{1}{5})^2 > B^2 - 1 \qquad \text{and} \quad (B+\tfrac{1}{5}) < 0 \\
&\Leftrightarrow \quad \tfrac{2}{5}B + \tfrac{1}{25} > -1 \qquad \text{and} \quad B < -\tfrac{1}{5} \\
&\Leftrightarrow \quad -\tfrac{1}{5} > B > -\tfrac{13}{5}.
\end{aligned}$$

However, if $B < -\frac{1}{5}$, then the condition that $\sqrt{B^2-1}$ is real, i.e. $|B| \geqslant 1$, implies the stronger condition $B \leqslant -1$. The condition that both roots are real *and* greater than $\frac{1}{5}$ is therefore

$$-\tfrac{13}{5} < B \leqslant -1$$

and the probability that both roots are real and greater than $\frac{1}{5}$ is

$$\Phi(-1) - \Phi(-2.6) = \Phi(2.6) - \Phi(1) = 0.9953 - 0.8413 = 0.1540.$$

The conditional probability that both roots are greater than $\frac{1}{5}$ *given* that they are real is

$$\begin{aligned}
\mathrm{P}\big(\text{both roots} > \tfrac{1}{5} \mid \text{both real}\big) \quad &= \quad \frac{\mathrm{P}\big(\text{ both roots} > \tfrac{1}{5} \textit{ and } \text{both roots real}\big)}{\mathrm{P}\big(\text{ both roots real}\big)} \\
&= \quad \frac{0.1540}{0.3174} = 0.485.
\end{aligned}$$

(ii) The sum of the roots is $-2B$, so we want the expectation of $|2B|$ given that $|B| \geqslant 1$, which is

$$\frac{\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{-1} (-2x)e^{-\frac{1}{2}x^2}\,\mathrm{d}x + \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{1}^{\infty} 2xe^{-\frac{1}{2}x^2}\,\mathrm{d}x}{\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{-1} e^{-\frac{1}{2}x^2}\,\mathrm{d}x + \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{1}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x} = \frac{2 \times \dfrac{1}{\sqrt{2\pi}} \times 2e^{-\frac{1}{2}}}{2\big(1 - \Phi(1)\big)} = 3.05.$$