

Problem 74: Breaking a stick

(✓✓)

A stick is broken at a point, chosen at random, along its length. Find the probability that the ratio, R , of the length of the shorter piece to the length of the longer piece is less than r , where r is a given positive number.

Find the probability density function for R , and calculate the mean and variance of R .

1999 Paper II

Comments

Continuous probability distributions often seem harder than discrete distributions, for no good reason: the concepts are the same, and in fact integrals are normally easier than sums.

Here, part of the difficulty is that you have to set up the problem yourself. You have a random variable with a known distribution (corresponding to the point at which the stick is broken), but you are interested in another random variable (the ratio of lengths) derived from the first. As always, it is best to work with the cumulative distribution functions rather than the probability density functions when deriving the distribution of the second random variable.

Solution to problem 74

Let the length of the stick be 2ℓ and let X be the length of the shorter piece of stick, so that $X \leq \ell$.

The random variable X is uniformly distributed on the interval $0 \leq x \leq \ell$, so

$$P(0 \leq X \leq x) = \frac{x}{\ell}.$$

Now $R = \frac{X}{2\ell - X}$, by definition, so

$$X = \frac{2\ell R}{1 + R}.$$

The cumulative distribution function for R is given by

$$P(R \leq r) = P\left(\frac{X}{2\ell - X} \leq r\right) = P\left(X \leq \frac{2\ell r}{1 + r}\right) = \frac{2\ell r}{(1 + r)\ell} = \frac{2r}{1 + r}.$$

Let's check that this satisfies the conditions for a cumulative distribution function: it should increase from 0 to 1 as r goes from its smallest value, which is 0, to its greatest value, which is 1. And it does, so that's OK.

The probability density function is the derivative of the cumulative distribution function:

$$\frac{d}{dr} \left(\frac{2r}{1 + r} \right) = \frac{2}{(r + 1)^2}.$$

Integrating to find $E(R)$ and $E(R^2)$ gives

$$E(R) = \int_0^1 \frac{2r}{(1 + r)^2} dr = 2 \ln 2 - 1; \quad E(R^2) = \int_0^1 \frac{2r^2}{(1 + r)^2} dr = 3 - 4 \ln 2,$$

and $\text{Var}(R) = (3 - 4 \ln 2) - (2 \ln 2 - 1)^2 = 2 - 4(\ln 2)^2$.

Post-mortem

Note the check we made to verify that the function we found for the cumulative distribution could actually be a cumulative distribution function, which is equivalent to checking that the probability density function integrates to 1.

We should also check that the final answers make sense. Putting numbers into a calculator gives $2 \ln 2 - 1 = 0.386$ for the expected value, which seems reasonable; at least it is less than 1. If you sketch the density function, you see that this could easily be its average value (do it!).

For the variance, I got 0.08, which means that the ratio is likely to be in the plausible range 0.15 to 0.31.

There are a variety of other stick-breaking problems, including a rather pleasing one about breaking the stick in two places and finding the probability that the pieces form a triangle. In this case, it matters how you break the stick: you could break it once and then break one of the pieces; or you could choose two points on the stick randomly at which to break it. For an interesting discussion, see the excellent cut-the-knot website: <http://www.cut-the-knot.org/Curriculum/Probability/TriProbability.shtml>