

Problem 70: Bodies in the fridge

(✓✓)

My two friends, who shall remain nameless, but whom I shall refer to as P and Q , both told me this afternoon that there is a body in my fridge. I'm not sure what to make of this, because P tells the truth with a probability of p , while Q (independently) tells the truth with a probability of only q . I haven't looked in the fridge for some time, so if you had asked me this morning, I would have said that there was just as likely to be a body in the fridge as not. Clearly, in view of what my friends have told me, I must revise this estimate. Explain carefully why my new estimate of the probability of there being a body in the fridge should be

$$\frac{pq}{1 - p - q + 2pq}.$$

I have now been to look in the fridge and there is indeed a body in it; perhaps more than one. It seems to me that only my enemy E_1 or my other enemy E_2 or (with a bit of luck) both E_1 and E_2 could be in my fridge, and this evening I would have judged these three possibilities equally likely. But tonight I asked P and Q separately whether E_1 was in the fridge, and they each said that she was. What should be my new estimate of the probability that both E_1 and E_2 are in my fridge?

Of course, I always tell the truth.

1987 Paper II

Comments

The most difficult part of this problem is unravelling the narrative! The first paragraph says essentially 'what is the probability that there is a body in the fridge, given that P and Q both say there is?'. It can therefore be tackled by the usual methods of conditional probability: tree diagrams, for example, or Bayes' theorem. All the other words in the first paragraph are there to tell you about the *a priori* probabilities³³ of the events, without knowledge of which the question above would be meaningless.

In the second paragraph, the situation becomes more complicated, but the method used for the first paragraph will still work.

In case you want to use it, here is the statement of Bayes' theorem, in its simplest form:

$$P(B|A) = \frac{P(B) \times P(A|B)}{P(A)}.$$

It is perhaps interesting to note that in 2002 it was decided that STEP questions should not be immoral (see Problem 67); but this seemed to be less of an issue in 1987.

³³ An *a priori* probability is a probability calculated theoretically (for example by using equally likely outcomes) or from existing information regarding a situation. It can then be revised using additional information, when it becomes an *a posteriori* probability.

Solution to problem 70

This problem can be solved using tree diagrams. A more sophisticated, but not necessarily better, method is to use Bayes' theorem.

Here, we take the events A and B to be

$$\begin{aligned} A &= P \text{ and } Q \text{ both say that there is a body in the fridge} \\ B &= \text{there is a body in the fridge} \end{aligned}$$

From the information given in the question, $P(B) = \frac{1}{2}$, so

$$P(B|A) = \frac{\frac{1}{2} \times pq}{P(A)}.$$

Now

$$\begin{aligned} P(A) &= P(\text{there is a body}) \times P(P \text{ and } Q \text{ both say there is}) \\ &\quad + P(\text{there is not a body}) \times P(\text{they both say there is}) \\ &= \frac{1}{2} \times pq + \frac{1}{2} \times (1-p)(1-q) \end{aligned}$$

which gives the required answer.

For the second paragraph, let

$$\begin{aligned} X &= P \text{ and } Q \text{ both say that } E_1 \text{ is in the fridge} \\ Y &= E_1 \text{ and } E_2 \text{ are in the fridge} \end{aligned}$$

There are three possibilities (since we know that there is at least one body in the fridge): only E_1 is in the fridge; only E_2 is in the fridge; and both E_1 and E_2 are in the fridge. These are given as equally likely, so the *a priori* probabilities are each $\frac{1}{3}$.

We want $P(Y|X)$, which by Bayes' theorem is

$$\frac{P(Y) \times P(X|Y)}{P(X)} = \frac{\frac{1}{3} \times pq}{P(X)}.$$

Now

$$\begin{aligned} P(X) &= P(\text{only } E_1 \text{ is in the fridge}) \times P(P \text{ and } Q \text{ told the truth}) \\ &\quad + P(\text{both } E_1 \text{ and } E_2 \text{ are in the fridge}) \times P(P \text{ and } Q \text{ both told the truth}) \\ &\quad + P(\text{only } E_2 \text{ is in the fridge}) \times P(P \text{ and } Q \text{ both lied}) \\ &= \frac{1}{3} \times pq + \frac{1}{3} \times pq + \frac{1}{3} \times (1-p)(1-q) \end{aligned}$$

so the answer is $\frac{pq}{1-p-q+3pq}$.

Post-mortem

Note how much more difficult it is when the answer is not given; when the question was originally set, most candidates arrived at the given answer to the first part but were not sufficiently confident to extend their method to the second paragraph: they received $\frac{8}{20}$ for their efforts.

The last line of the question is not entirely frivolous; if I may have lied about what my friends answered when I asked them if there is a body in the fridge, the problem becomes difficult. However, my claim to be truthful is vacuous (it tells you nothing) because I may be lying. Contrast with the statement 'I am lying', which is inconsistent.