

Problem 69: Lottery

(✓✓)

The national lottery of Ruritania is based on the positive integers from 1 to N , where N is very large and fixed. Tickets cost £1 each. For each ticket purchased, the punter (i.e. the purchaser) chooses a number from 1 to N . The winning number is chosen at random, and the jackpot is shared equally amongst those punters who chose the winning number.

A syndicate decides to buy N tickets, choosing every number once to be sure of winning a share of the jackpot. The total number of tickets purchased in this draw is $3.8N$ and the jackpot is £ W . Assuming that the non-syndicate punters choose their numbers independently and at random, find the most probable number of winning tickets and show that the expected net loss of the syndicate is approximately

$$N - \frac{5(1 - e^{-2.8})}{14} W.$$

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Comments

This is a Binomial distribution problem: the probability that n out of m punters choose the winning ticket is

$$\binom{m}{n} p^n q^{m-n}$$

where here $m = 2.8N$, $p = 1/N$ and $q = 1 - p$. It is clear that an approximation to the Binomial distribution is expected (for example, the question uses the word ‘approximately’ and you have to think about how an approximation might arise); and the presence of the exponential in the given result gives a pretty broad hint that it should be a Poisson distribution — the use of which has to be justified. One can expect the Poisson approximation to work when the number of trials (call it m) is large (e.g. $m > 150$) and when $np \approx npq$, i.e. the mean is roughly equal to the variance (since these are equal for the Poisson distribution) — so $q \approx 1$.

You can’t find the most probable number of winning tickets by differentiation (unless you fancy differentiating the factorial $x!$); instead, you must look at the ratios of consecutive terms and see when these turn from being greater than one to less than one.

Solution to problem 69

Let X be the random variable whose value is the number of winning tickets out of the $2.8N$ tickets purchased by the non-syndicate punters. Then $X \sim \text{Poisson}(2.8)$.

Let $p_j = P(X = j)$. Then

$$p_j = \frac{(2.8)^j e^{-2.8}}{j!} \Rightarrow \frac{p_{j+1}}{p_j} = \frac{2.8}{j+1}.$$

This fraction is greater than 1 if $j < 1.8$, so the most probable number of winning tickets by non-syndicate punters is 2. Overall (including the ticket bought by the syndicate), the most probable number of winning tickets is 3, which is very plausible.

The expected winnings of the syndicate is

$$\begin{aligned} \left(Wp_0 + \frac{1}{2}Wp_1 + \frac{1}{3}Wp_2 + \dots \right) &= e^{-2.8}W \left(1 + \frac{1}{2} \times \frac{1}{1!}(2.8) + \frac{1}{3} \times \frac{1}{2!}(2.8)^2 + \dots \right) \\ &= e^{-2.8}W \frac{e^{2.8} - 1}{2.8}, \end{aligned}$$

so the expected loss is as given (note that $2.8 = \frac{14}{5}$).

Post-mortem

This is rather interesting. You might perhaps have considered whether it would be worth borrowing money to buy every single lottery combination, in order to win a share of the jackpot. Clearly, the people who run the lottery have to think about this sort of thing.

Suppose the number of tickets sold, excluding the N that we plan to buy, is expected to be kN . Suppose also that a fraction α of the total is paid out in the jackpot. Then, setting $2.8 = k$ and $W = \alpha(k+1)N$, the expected loss formula given in the question becomes

$$N \left(1 - \frac{1 - e^{-k}}{k} \alpha(k+1) \right).$$

If k is very small (take $k = 0$), we lose $N(1 - \alpha)$ (obviously). If k is large, $k = 3$ say, the exponential can be ignored and we lose $N(1 - \alpha(k+1)/k)$. If $k \gg 1$, this becomes $N(1 - \alpha)$ again. In between, there is a value of k that, for each fixed α , gives a minimum loss (which may be a gain if α is close to 1).

Note how informative it is to have k rather than 2.8; the numerical value was chosen in the question to model roughly that lottery system in the UK.

Having gone back to this solution after a break, I am now wondering about the use of the Poisson approximation. Of course, the set-up (large m , small p) begs us to approximate, but did we need to? Certainly not for the first result, since we can just as well look at the ratio of two terms of the Binomial distribution as at two terms of the Poisson distribution. Try it; the result is of course the same.

The second part is more difficult. What we want is the expectation of $1/(n+1)$, and this turns out to be a difficult sum using the Binomial distribution (in fact, it can only be expressed in terms of a hypergeometric function, which would then have to be approximated to get a less obscure answer).

It seems to me that the solution above is therefore a bit unsatisfactory. It would surely have been better to work with the exact distribution until it was necessary to approximate, even though one knows that the approximation is so good that the answers would be the same. We are, after all, mathematicians and not engineers.