Problem 68: Breaking plates ($\checkmark$)

Four students, one of whom is a mathematician, take turns at washing up over a long period of time. The number of plates broken by any student in this time obeys a Poisson distribution, the probability of any given student breaking n plates being $e^{-\lambda}\lambda^n/n!$ for some fixed constant λ, independent of the number of breakages by other students. Given that five plates are broken, find the probability that three or more were broken by the mathematician.

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Comments

The way this is set up, it is largely a counting exercise (but see the post-mortem). To start with, you work out the probability of 5 breakages, then follow that with the probability that the mathematician broke 3 or more plates. You need to calculate the number of ways that 5 plates can be shared amongst four students, for which you have to consider each *partition* of the number 5 and the number of different ways it can arise.

Solution to problem 68

First we work out the probability of 5 breakages.

Let $\mathrm{P}(5,0,0,0)$ denote the probability that student A breaks 5 plates and students B, C and D break 0 plates. Then

$$\mathrm{P}(5,0,0,0) = (\text{Prob. of breaking } 5) \times (\text{Prob. of breaking } 0)^3 = \frac{\mathrm{e}^{-\lambda}\lambda^5}{5!}\left(\mathrm{e}^{-\lambda}\right)^3 = \frac{\lambda^5\mathrm{e}^{-4\lambda}}{5!}.$$

The probability that one student breaks all the plates is $4 \times \lambda^5\mathrm{e}^{-4\lambda}/5!$, the factor 4 because it could be any one of the four students.

Let $\mathrm{P}(4,1,0,0)$ denote the probability that student A breaks 4 plates, student B breaks 1 plate and students C and D break 0 plates. Then

$$\begin{aligned}\mathrm{P}(4,1,0,0) &= (\text{Prob. of breaking } 4) \times (\text{Prob. of breaking } 1) \times (\text{Prob. of breaking } 0)^2 \\ &= \frac{\mathrm{e}^{-\lambda}\lambda^4}{4!}\,\frac{\mathrm{e}^{-\lambda}\lambda}{1!}\left(\mathrm{e}^{-\lambda}\right)^2 = \frac{\lambda^5\mathrm{e}^{-4\lambda}}{4!}\,.\end{aligned}$$

The probability that any one student breaks 4 plates and any other student breaks 1 plate is therefore $4 \times 3 \times \lambda^5\mathrm{e}^{-4\lambda}/4!\,$.

Considering $\mathrm{P}(3,2,0,0)$, $\mathrm{P}(3,1,1,0)$, $\mathrm{P}(2,2,1,0)$ and $\mathrm{P}(2,1,1,1)$ in turn, together with the number of different ways these probabilities can occur, shows that the probability of 5 breakages is

$$\lambda^5\mathrm{e}^{-4\lambda}\left(\frac{4}{5!} + \frac{12}{4!\,1!} + \frac{12}{3!\,2!} + \frac{12}{3!\,1!\,1!} + \frac{12}{2!\,2!\,1!} + \frac{4}{2!\,1!\,1!\,1!}\right) = \lambda^5\mathrm{e}^{-4\lambda}\left(\frac{1024}{5!}\right)\,.$$

The probability that the mathematician (student A, say) breaks 3 or more is $\mathrm{P}(5,0,0,0) + 3\mathrm{P}(4,1,0,0) + 3\mathrm{P}(3,2,0,0) + 3\mathrm{P}(3,1,1,0)$, i.e.

$$\lambda^5\mathrm{e}^{-4\lambda}\left(\frac{4}{5!} + \frac{3}{4!\,1!} + \frac{3}{3!\,2!} + \frac{4}{3!\,1!\,1!}\right) = \lambda^5\mathrm{e}^{-4\lambda}\left(\frac{106}{5!}\right)\,.$$

The probability that the mathematician breaks 3 or more, given that 5 are broken, is therefore $106/1024\,$.

Post-mortem

If you did the question by the method suggested above, you will probably be wondering about the answer: why is it independent of λ; and why is the denominator 4^5? You will quickly decide that the Poisson distribution was a red herring (though I promise you that it was not an intentional herring), since there is no trace of the distribution in the answer.

The denominator is pretty suggestive. A completely different approach is as follows, It is clear that the probability that the mathematician breaks any given plate is $\frac{1}{4}$. The probability that he or she breaks k plates is therefore binomial:

$$\binom{5}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{5-k}$$

and adding the cases $k = 3$, $k = 4$ and $k = 5$ gives rather rapidly

$$\frac{90 + 15 + 1}{1024}\,.$$

For the method given in the solution, we could substitute P_k for $\mathrm{e}^{-\lambda}\lambda^k/k!$ (as the probability of breaking k plates) and the calculation would work just the same, with all the P_ks disappearing; try it.