

Problem 61: Kinematics of rotating target

(✓✓)

An automated mobile dummy target for gunnery practice is moving anti-clockwise around the circumference of a large circle of radius R in a horizontal plane at a constant angular speed ω . A shell is fired from O , the centre of this circle, with initial speed V and angle of elevation α .

- (i) Show that if $V^2 < gR$, then no matter what the value of α , or what vertical plane the shell is fired in, the shell cannot hit the target.
- (ii) Assume now that $V^2 > gR$ and that the shell hits the target, and let β be the (positive) angle through which the target rotates between the time at which the shell is fired and the time of impact. Show that β satisfies the equation

$$g^2\beta^4 - 4\omega^2V^2\beta^2 + 4R^2\omega^4 = 0.$$

Deduce that there are exactly two possible values of β .

- (iii) Let β_1 and β_2 be the possible values of β and let P_1 and P_2 be the corresponding points of impact. By considering the quantities $(\beta_1^2 + \beta_2^2)$ and $\beta_1^2\beta_2^2$, or otherwise, show that the linear distance between P_1 and P_2 is

$$2R \sin \left(\frac{\omega}{g} \sqrt{V^2 - Rg} \right).$$

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Comments

The rotation of the target is irrelevant for the first part, which contravenes the setters' rule of not introducing information before it is required. In this case, it seemed better to describe the set-up immediately — especially as you are asked for a familiar result.

Remember, when you are considering the roots of the quartic (which is really a quadratic in β^2), that the question gives $\beta > 0$.

The hint in the last paragraph ('by considering ...') is supposed to direct you towards the relation between the coefficients in a quadratic equation and the sum and product of the roots; otherwise, you get into some pretty heavy algebra.

Solution to problem 61

(i) This part is about the range of the gun. If the shell lands at distance x , then

$$x = (V \cos \alpha)t, \quad 0 = (V \sin \alpha)t - \frac{1}{2}gt^2.$$

Eliminating t gives $xg = V^2 \sin 2\alpha$. The range r is the largest value of x , which is given by $rg = V^2$. The shell cannot reach the target (for any angle of elevation) if $R > r$, so the shell cannot hit its target if $V^2 < Rg$.

(ii) The time of flight is $R/(V \cos \alpha)$, and this must also equal the time for the target to rotate through β , i.e. β/ω . Thus $\cos \alpha = R\omega/(\beta V)$. Substituting this into the range equation $Rg = 2V^2 \sin \alpha \cos \alpha$ gives

$$Rg = 2V^2 \frac{R\omega}{\beta V} \left(1 - \frac{R^2\omega^2}{\beta^2 V^2}\right)^{\frac{1}{2}}.$$

Squaring both sides of the equation, and simplifying, leads to the given quadratic in β^2 . Solving the quadratic using the quadratic formula gives

$$g^2\beta^2 = 2\omega^2 \left(V^2 \pm \sqrt{V^4 - g^2 R^2}\right).$$

Since $V^2 > gR$, both the values of β^2 roots are real and positive, but they lead to only two relevant values of β since we only want positive values.

(iii) The linear distance between the two points of impact is $2R \sin \frac{1}{2}(\beta_2 - \beta_1)$. Now $\beta_1^2 + \beta_2^2 = 4\omega^2 V^2/g^2$ and $\beta_1^2 \beta_2^2 = 4R^2 \omega^4/g^2$ (sum and product of the roots of the quadratic), so

$$(\beta_2 - \beta_1)^2 = \beta_2^2 + \beta_1^2 - 2\beta_1\beta_2 = \frac{4\omega^2 V^2}{g^2} - \frac{4R\omega^2}{g},$$

which leads to the given result.

Post-mortem

Why do we get two values for β ? In order to hit the large circle, if it is in range, we can fix any speed of projection and choose the angle of projection appropriately. In general, there are two possible angles of projection, one above 45° and one below 45° (satisfying $\sin 2\alpha = Rg/V^2$).

However, in this question, we want to hit the circle essentially at a given time (when the target is at the landing place), which means that $V \cos \alpha$ is fixed, though of course it is a little more complicated because the position of the target is fixed but unknown. But still, the two values of β , corresponding to two different angles of elevation, emerge in the same way.

Did you notice the slight inaccuracy in the question? The final 'distance' could be negative (if for example $\pi g < \omega \sqrt{V^2 - Rg} < 2\pi g$). Mod signs are needed.