Problem 60: Newton's cradle (✓✓)

N particles P_1, P_2, P_3, …, P_N with masses m, qm, q^2m, … , $q^{N-1}m$, respectively, are at rest at distinct points along a straight line in gravity-free space. The particle P_1 is set in motion towards P_2 with velocity V and in every subsequent impact the coefficient of restitution is e, where $0 < e < 1$.

(i) Show that after the first impact the velocities of P_1 and P_2 are

$$\left(\frac{1-eq}{1+q}\right)V \quad \text{and} \quad \left(\frac{1+e}{1+q}\right)V,$$

respectively.

(ii) Show that, if $q \leqslant e$, then there are exactly $N-1$ impacts.

(iii) Show further that, if $q = e$, then the total loss of kinetic energy after all impacts have occurred is

$$\tfrac{1}{2}me\big(1-e^{N-1}\big)V^2.$$

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Comments

This situation models the toy called 'Newton's Cradle' which consists of four or more heavy metal balls suspended from a frame so that they can swing. At rest, they are in contact in a line. When the first ball is raised and let swing, there follows a rather pleasing pattern of impacts. In this case, the coefficient of restitution is nearly 1 and the balls all have the same mass, so, as the first displayed formula shows, the impacting ball is reduced to rest by the impact. At the first swing of the ball, nothing happens except that the first ball is reduced to rest and the last ball swings away. Note that this is consistent with the balls being separated by a very small amount; what actually happens is that the ball undergoes a small elastic deformation at the impact, and the impulse takes a small amount of time to be transmitted across the ball to the next ball.

Solution to problem 60

(i) Let V_1 and V_2 be the velocities of P_1 and P_2 after the first collision. Using conservation of momentum and Newton's law of impact at the first collision results in the equations

$$mV = mV_1 + mqV_2\,, \quad V_1 - V_2 = -eV\,,$$

so

$$V_1 = \left(\frac{1-eq}{1+q}\right)V \quad \text{and} \quad V_2 = \left(\frac{1+e}{1+q}\right)V\,.$$

(ii) Note that V_1 is positive since $eq \leqslant e^2 < 1$. Conserving momentum and using Newton's law (and doing a bit of algebra) shows that the speeds $\widehat{V}_2$ and V_3 of P_2 and P_3 after the next collision are given by

$$\widehat{V}_2 = \left(\frac{1-eq}{1+q}\right)V_2 = \left(\frac{1-eq}{1+q}\right)\left(\frac{1+e}{1+q}\right)V \quad \text{and} \quad V_3 = \left(\frac{1+e}{1+q}\right)V_2 = \left(\frac{1+e}{1+q}\right)^2 V. \qquad (*)$$

Note that $\widehat{V}_2 \geqslant V_1$ so there is no further collision between P_1 and P_2. Applying this argument at each collision shows that there are exactly $N-1$ collisions: P_1 with P_2; P_2 with P_3; etc.

(iii) The speed of P_k after it has hit P_{k+1} is (by extending $(*)$)

$$\left(\frac{1-eq}{1+q}\right)\left(\frac{1+e}{1+q}\right)^{k-1} V,$$

and the speed of P_{k+1} after this collision and before it hits P_{k+2} is

$$\left(\frac{1+e}{1+q}\right)^{k} V.$$

If $q = e$,

$$V_1 = \left(\frac{1-e^2}{1+e}\right)V = (1-e)V$$

and similarly the final speeds of $P_2, \ldots, P_{N-1}$ are all $(1-e)V$. The final speed of P_N is V. Thus the final total kinetic energy is

$$\begin{aligned}\tfrac{1}{2}(m + mq + \cdots + mq^{N-2})[(1-e)V]^2 + \tfrac{1}{2}mq^{N-1}V^2 \quad &= \quad \tfrac{1}{2}m\,\frac{1-e^{N-1}}{1-e}\,(1-e)^2V^2 + \tfrac{1}{2}me^{N-1}V^2 \\ &= \quad \tfrac{1}{2}m(1-e+e^N)V^2,\end{aligned}$$

(replacing all the q's with e's and summing the geometric progression). Thus the loss of kinetic energy is

$$\tfrac{1}{2}mV^2 - \tfrac{1}{2}m(1-e+e^N)V^2,$$

as required.

Post-mortem

There are two tricky aspects to these multiple collision questions. First there is the matter of notation. Above, I have used a hat for the second collision of a particle ($\widehat{V}_2$), retaining the subscript for labelling particles. That works, but if P_2 undergoes another collision, entailing a double hat, it starts getting messy. You could use a different letter, but that gets confusing. The only good method is to use a double subscript: $V_{m,n}$ is the velocity of the mth particle after the nth collision. But that is a sledgehammer for this nut of a question.

The other tricky aspect is getting the signs right. I always think of velocity (not speed), and it is always positive for particles travelling to the right. I also use common sense to check each equation! As usual, a good diagram of each collision is essential.