Problem 58: Fielding (✓✓)

In a game of cricket, a fielder is perfectly placed to catch a ball. She watches the ball in flight and takes the catch just in front of her eye. The angle between the horizontal and her line of sight at a time t after the ball is struck is θ. Show that $\dfrac{\mathrm{d}}{\mathrm{dt}}(\tan\theta)$ is constant during the flight.

The next ball is also struck in the direction of the fielder but at a different velocity. In order to be perfectly placed to catch the ball, the fielder runs at constant speed towards the batsman. Assuming that the ground is horizontal, show that again $\dfrac{\mathrm{d}}{\mathrm{dt}}(\tan\theta)$ is constant during the flight.

1998 Paper II

Comments

As with all the very best questions, nine-tenths of this question is submerged below the surface. It uses the deepest properties of Newtonian dynamics, and a good understanding of the subject makes the question completely transparent. However, it can still be done without too much trouble by a straightforward approach, in which case the difficulty lies only in setting it up for yourself.

The cleverness of this question lies in its use of two fundamental invariances of Newton's second law:

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = m\mathbf{g}. \tag{†}$$

The first is invariance under *time reflection symmetry*, which arises because equation (†) is not affected by the transformation $t \to -t$. This means that any given solution can be replaced by one where the projectile goes back along the trajectory, i.e. time runs backwards.

The second is invariance under what are called *Galilean transformations*. Equation (†) is also invariant under the transformation $\mathbf{x} \to \mathbf{x} + \mathbf{v}t$, where $\mathbf{v}$ is an arbitrary constant velocity. This means that we can solve the equation in a frame that moves with constant speed.

Solution to problem 58

We take a straightforward approach. Let the height above the fielder's eye level at which the ball is struck be h. Let the speed at which the ball is struck be u and the angle which the trajectory of the ball initially makes with the horizontal be α.

Then, taking x to be the horizontal distance of the ball at time t from the point at which the ball was struck and y to be the height of the ball at time t above the fielder's eye level[31], we have

$$x = (u\cos\alpha)t\ , \quad y = h + (u\sin\alpha)t - \tfrac{1}{2}gt^2\ .$$

Let d be the horizontal distance of the fielder from the point at which the ball is struck, and let T be the time of flight of the ball. Then

$$d = (u\cos\alpha)T\ , \quad 0 = h + (u\sin\alpha)T - \tfrac{1}{2}gT^2 \qquad (*)$$

and

$$\begin{aligned}
\tan\theta = \frac{y}{d-x} &= \frac{h + (u\sin\alpha)t - \frac{1}{2}gt^2}{d - (u\cos\alpha)t} \\
&= \frac{-(u\sin\alpha)T + \frac{1}{2}gT^2 + (u\sin\alpha)t - \frac{1}{2}gt^2}{(u\cos\alpha)T - (u\cos\alpha)t} && \text{(using $*$)} \\
&= \frac{-(u\sin\alpha)(T-t) + \frac{1}{2}g(T^2 - t^2)}{(u\cos\alpha)(T-t)} \\
&= \frac{-u\sin\alpha + \frac{1}{2}g(T+t)}{u\cos\alpha}. && \text{(cancelling the factor $(T-t)$)}
\end{aligned}$$

This last expression is a polynomial of degree one in t, so its derivative is constant, as required.

For the second part, let l be the distance from the fielder's original position to the point at which she catches the ball. Then $l = vT$ and

$$\tan\theta = \frac{y}{(l-vt)+d-x} = \frac{y}{v(T-t)+d-x} = \frac{-u\sin\alpha + \frac{1}{2}g(T+t)}{v + u\cos\alpha}$$

cancelling the factor of $(T-t)$ as before. This again has constant derivative.

Post-mortem

The invariance mentioned in the comments section above can be used to answer the question almost without calculation.

Using time reflection symmetry to reverse the trajectory shows that the batsman is completely irrelevant: it only matters that the fielder caught a ball. We just think of the ball being projected from the fielder's hands (the time-reverse of a catch). Taking her hands as the origin of coordinates, and using u to denote the projection (i.e. the catching) speed of the ball and α to be the angle of projection (i.e. the final value of θ), we have $y = (u\sin\alpha)t - \frac{1}{2}gt^2$, $x = (u\cos\alpha)t$ and $\tan\theta = (u\sin\alpha - \frac{1}{2}gt)/u\cos\alpha$. The first derivative of this expression is constant, as before.

We use the Galilean transformation for the second part of the question. Instead of thinking of the fielder running with constant speed v towards the batsman, we can think of the fielder being stationary and the ball having an additional horizontal speed of v. The situation is therefore not changed from that of the first part of the question, except that $u\cos\theta$ should be replaced by $v + u\cos\theta$.

[31] Draw a diagram! I would, but there isn't enough room on the page.