

Problem 55: Hank's gold mine

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Hank's Gold Mine has a very long vertical shaft of height l . A light chain of length l passes over a small smooth light fixed pulley at the top of the shaft. To one end of the chain is attached a bucket A of negligible mass and to the other a bucket B of mass m .

The system is used to raise ore from the mine as follows. When bucket A is at the top it is filled with mass $2m$ of water and bucket B is filled with mass λm of ore, where $0 < \lambda < 1$. The buckets are then released, so that bucket A descends and bucket B ascends. When bucket B reaches the top both buckets are emptied and released, so that bucket B descends and bucket A ascends. The time to fill and empty the buckets is negligible. Find the time taken from the moment bucket A is released at the top until the first time it reaches the top again.

This process goes on for a very long time. Show that, if the greatest amount of ore is to be raised in that time, then λ must satisfy the condition $f'(\lambda) = 0$ where

$$f(\lambda) = \frac{\lambda(1-\lambda)^{1/2}}{(1-\lambda)^{1/2} + (3+\lambda)^{1/2}}.$$

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Comments

One way of working out the acceleration of a system of two masses connected by a light string passing over a pulley is to write down the equation of motion of each mass, bearing in mind that the force due to tension will be the same for each mass (it cannot vary along the string, because then the acceleration of some portion of the *massless* string would be infinite). Then you eliminate the tension.

Alternatively, you can use the equation of motion of the system of two joined masses. The system has inertial mass equal to the sum of the masses (because both masses must accelerate equally) but gravitational mass equal to the difference of the masses (because the gravitational force on one mass cancels, partially, the gravitational force on the other), so the equation of motion is just (Newton's law of motion)

$$(m_1 + m_2)a = (m_1 - m_2)g.$$

Solution to problem 55

When bucket A ascends, the acceleration is g .

For bucket A 's downward journey, at acceleration a , the equations of motion for bucket A and bucket B , respectively, are

$$-T + 2mg = 2ma, \quad T - (1 + \lambda)mg = (1 + \lambda)ma,$$

where T is the tension in the rope. Eliminating T gives so $a = \frac{1 - \lambda}{3 + \lambda} g$.

The time of descent (using $l = \frac{1}{2}at^2$) is $\sqrt{2l/a}$ and the time of ascent is $\sqrt{2l/g}$. The total time required for one complete cycle is therefore

$$\sqrt{\frac{2l}{g}} \left(1 + \sqrt{\frac{3 + \lambda}{1 - \lambda}} \right).$$

Call this t . The number of round trips in a long time t_{long} is t_{long}/t so the amount of ore lifted in time t_{long} is $\lambda m t_{\text{long}}/t$.

To maximise this, we have to maximise λ/t with respect to λ , and λ/t is exactly the $f(\lambda)$ given. Note that $f(0) = 0$ (which makes sense because no ore is raised if $\lambda = 0$), and $f(1) = 0$ (which also makes sense because the buckets don't move on the raising stage if $\lambda = 1$). That means the greatest value of $f(\lambda)$ must occur at a value of λ in the range $0 < \lambda < 1$ at which $f'(\lambda) = 0$.

Post-mortem

You may have wondered why, in the last part, the question says that the process goes on for a very long time. The reason for this is that when you maximise what I have called $\lambda m t_{\text{long}}/t$ the result may not correspond to a complete number of cycles. If you stop in mid-cycle, you raise no ore from that cycle so a calculus maximisation of a continuous function is not the right method. However, if the process continues for many cycles, the contribution from the last cycle becomes negligible, and a calculus maximisation becomes appropriate.

You may also wonder why you were not asked to find the maximising value of λ . If you are feeling exceptionally energetic, you could try to solve $f'(\lambda) = 0$. This will eventually lead you to the rather discouraging quartic equation

$$\lambda^4 + 4\lambda^3 + 2\lambda^2 - 8\lambda + 3 = 0.$$

You will have done a lot of tedious work to obtain an equation that cannot be solved without either a formula for the solutions of a general quartic equation or a computer. There are in fact two real solutions of this equation, $\lambda = 0.528$ and $\lambda = 0.704$.