

Problem 54: How did the chicken cross the road?

(✓✓)

A single stream of cars, each of width a and exactly in line, is passing along a straight road of breadth b with speed V . The distance between successive cars (i.e. the distance between the back of one car and the front of the following car) is c .

A chicken crosses the road in safety at a constant speed u in a straight line making an angle θ with the direction of traffic. Show that

$$u \geq \frac{Va}{c \sin \theta + a \cos \theta} . \quad (*)$$

Show also that if the chicken chooses θ and u so that she crosses the road at the least possible (constant) speed, she crosses in time

$$\frac{b}{V} \left(\frac{c}{a} + \frac{a}{c} \right) .$$

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Comments

I like this question because it relates to (an idealised version of) a situation we have probably all thought about. Once you have visualised it, there are no great difficulties. As usual, you have to be careful with the inequalities, though it turns out here that there is no danger of dividing by a negative quantity.

Solution to problem 54

The easiest way to think about this problem is to consider the cars to be stationary and the velocity of the chicken to be $(u \cos \theta - V, u \sin \theta)$. Then the diagrams are very easy to visualise.

Let t be the time taken to cross the distance a in which the chicken is at risk. Then $a = ut \sin \theta$.

For safety, the chicken must choose $ut \cos \theta + c \geq Vt$: equality here occurs when the chicken starts at the near-side rear of one car and just avoids being hit by the far-side front of the next car.

Eliminating t from these two equations gives the required inequality:

$$\begin{aligned} ut \cos \theta &\geq Vt - c \\ \implies (u \cos \theta - V)t &\geq -c \\ \implies (u \cos \theta - V) \frac{a}{u \sin \theta} &\geq -c \\ \implies au \cos \theta - aV &\geq -cu \sin \theta \\ \implies u(c \sin \theta + a \cos \theta) &\geq aV \end{aligned}$$

which is the required result.

For a given value of θ , the minimum speed satisfies

$$u(c \sin \theta + a \cos \theta) = aV.$$

The smallest value of this u is therefore obtained when $c \sin \theta + a \cos \theta$ is largest. This can be found by calculus (regard it as a function of θ and differentiate: the maximum occurs when $\tan \theta = c/a$) or by trigonometry:

$$c \sin \theta + a \cos \theta = \sqrt{a^2 + c^2} \cos(\theta - \tan^{-1}(c/a))$$

so the maximum value is $\sqrt{a^2 + c^2}$ and it occurs when $\tan \theta = c/a$.

The time of crossing is

$$\frac{b}{u \sin \theta} = \frac{b(c \sin \theta + a \cos \theta)}{Va \sin \theta} = \frac{b(c + a \cot \theta)}{Va} = \frac{b(c + a^2/c)}{Va}.$$

Post-mortem

There is another inequality besides (*) that you might have noticed. If $u \cos \theta > V$ (so the chicken moves faster than the cars — a bit unlikely unless the chicken is trying to cross the M25), the chicken should start her run at the *front* nearside of a car and must not collide with the car *ahead*. This requires $(u \cos \theta - V)t \leq c$, so

$$u(-c \sin \theta + a \cos \theta) \leq aV.$$

If $(-c \sin \theta + a \cos \theta) < 0$, this places no constraint on u . But if $(-c \sin \theta + a \cos \theta) > 0$, then

$$u \leq \frac{aV}{-c \sin \theta + a \cos \theta}.$$

In both cases, the inequality (*) overleaf does not apply. This is clearly not the situation envisaged by the examiners, and probably not by any of the candidates either, but still it should have been catered for in the wording of the question.