Problem 52: Snowploughing (✓✓✓)

Two identical snowploughs plough the same stretch of road. The first starts at a time t_1 seconds after it starts snowing, and the second starts from the same point $t_2 - t_1$ seconds later, going in the same direction. Snow falls so that the depth of snow increases at a constant rate of k ms^{-1}. The speed of each snowplough is ak/z ms^{-1} where z is the depth (in metres) of the snow it is ploughing and a is a constant. Each snowplough clears all the snow. Show that the time t at which the second snowplough has travelled a distance x metres satisfies the equation

$$a\frac{\mathrm{d}t}{\mathrm{d}x} = t - t_1 e^{x/a}. \qquad (\dagger)$$

Hence show that the snowploughs will collide when they have travelled $a(t_2/t_1 - 1)$ metres.

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Comments

There is something exceptionally beautiful about this question, but it is hard to identify exactly what it is; seeing this question for the first time makes even hardened mathematicians smile with pleasure.

There is a modelling element to it: you have to set up equations from the information given in the text. The first equation you need is a simple first order differential equation to find the time taken by the first snowplough to travel a distance x. The corresponding equation for the second snowplough is a bit more complicated, because the depth of snow at any point depends on the time at which the first snowplough reached that point, clearing the snow.

The differential equation ($\dagger$) can be solved using an integrating factor. However, the equation which arises naturally at this point is one involving $\dfrac{\mathrm{d}x}{\mathrm{d}t}$, which cannot (apparently) be solved by any means. It is the rather good trick of turning the equation upside down (regarding t as a function of x instead of x as a function of t) that allows the problem to be solved so neatly. Apologies if you haven't come across integrating factors for first order differential equations; but they are really not difficult—you can look online and find an easily understandable explanation.

You won't surprised to learn that there is a generalisation to n identical snowploughs . . . .

Solution to problem 52

Suppose that the first snowplough reaches a distance x at time T after it starts snowing. Then the depth of snow it encounters is kT and its speed is therefore $ak/(kT)$, i.e. a/T. The equation of motion of the first snowplough is

$$\frac{\mathrm{d}x}{\mathrm{d}T} = \frac{a}{T}.$$

Integrating both sides with respect to T gives

$$x = a \ln T + \text{constant of integration.}$$

We know that $T = t_1$ when $x = 0$ (the snowplough started t_1 seconds after the snow started), so

$$x = a \ln T - a \ln t_1.$$

This can be rewritten as

$$T = t_1 e^{x/a}\,.$$

When the second snowplough reaches x at time t, snow has been falling for a time $t - T$ since it was cleared by the first snowplough, so the depth at time t is $k(t - T)$ metres, i.e. $k(t - t_1 e^{x/a})$ metres. Thus the equation of motion of the second snowplough is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{ak}{k(t - t_1 e^{x/a})}\,.$$

Now we use the standard result (a special case of the chain rule)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1\bigg/\frac{\mathrm{d}t}{\mathrm{d}x}$$

to obtain the required equation (†).

Multiplying by $e^{-x/a}$ (an *integrating factor*) and rearranging gives

$$e^{-x/a}\frac{\mathrm{d}t}{\mathrm{d}x} - \frac{e^{-x/a}t}{a} = -\frac{t_1}{a} \qquad \text{i.e.} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(te^{-x/a}\right) = -\frac{t_1}{a}$$

which integrates to

$$te^{-x/a} = -\frac{t_1}{a}x + \text{constant of integration.}$$

Since the second snowplough started ($x = 0$) at time t_2, the constant of integration is just t_2 and the solution is

$$t = (t_2 - t_1 x/a)e^{x/a}.$$

The snowploughs collide when they reach the same position at the same time. Let this position be $x = X$. Then

$$T = t \Longrightarrow t_1 e^{X/a} = (t_2 - t_1 X/a)e^{X/a},$$

so X is given by

$$t_1 = (t_2 - t_1 X/a).$$

This is equivalent to the given formula.