

## Problem 51: Spherical loaf

(✓✓✓)

**Note:** In this question slices of bread are eaten in order, starting at one end of the loaf.

A spherical loaf of bread is cut into parallel slices of equal thickness. Show that, after any number of the slices have been eaten, the area of crust remaining is proportional to the number of slices remaining.

A European ruling decrees that a parallel-sliced spherical loaf can only be referred to as ‘crusty’ if the ratio of volume  $V$  (in cubic metres) of bread remaining to area  $A$  (in square metres) of crust remaining after any number of slices have been eaten satisfies  $V < A$ . Show that the radius of a crusty parallel-sliced spherical loaf must be less than  $2\frac{2}{3}$  metres.

[The area  $A$  and volume  $V$  formed by rotating a curve in the  $x$ - $y$  plane round the  $x$ -axis from  $x = a - t$  to  $x = a$  are given by

$$A = 2\pi \int_{a-t}^a y \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)^{\frac{1}{2}} dx, \quad V = \pi \int_{a-t}^a y^2 dx . ]$$

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## Comments

The first result (the mathematical result, I mean, not the European ruling which I invented) came as a bit of a surprise to me — though no doubt it is well known. I wondered if it was the only surface of revolution with this property. You might like to think about this.

Don’t be distracted by the use of the word ‘slices’; since the thickness of the slices is not given, it is clear that you are supposed to think in terms of the continuous distance along the loaf rather than the number of slices.

For the last part, you will need to minimise a ratio as a function of  $t$  (the ‘length’ of loaf remaining). To find the ratio you have to do a couple of integrals. It is this ‘multi-stepping’ that makes the problem difficult (and very different from typical school-level questions) rather than any individual step.

The note at the beginning of the question is new. Seventeen years after the question was set two people wrote, independently, to say the solution was wrong and the question was wrongish. If you eat the slices out of order you may eat the middle slice first which gives the weaker result  $r < 3$  for the last part. The first paragraph is still OK, because each slice has the same area of crust.

Volumes of revolution are no longer in the A level Mathematics specifications so have been moved from the STEP 1 to the STEP 3 specification.

## Solution to problem 51

The first thing we need is an equation for the surface of a spherical loaf. The obvious choice, especially given the hint at the bottom of the question, is the circle  $x^2 + y^2 = a^2$  in the plane  $z = 0$ , rotated about the  $x$ -axis.

If the loaf is cut at a distance  $t$  from the end  $x = a$ , and the portion from  $x = -a$  to  $x = a - t$  is eaten, then the area remaining is

$$\begin{aligned} 2\pi \int_{a-t}^a y \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)^{\frac{1}{2}} dx &= 2\pi \int_{a-t}^a (a^2 - x^2)^{\frac{1}{2}} \left( 1 + \left( \frac{-x}{(a^2 - x^2)^{\frac{1}{2}}} \right)^2 \right)^{\frac{1}{2}} dx \\ &= 2\pi \int_{a-t}^a a \, dx \\ &= 2\pi at. \end{aligned} \quad (*)$$

This is proportional to the length  $t$  of remaining loaf, so proportional to the number of slices remaining (if the loaf is evenly sliced).

The remaining volume is

$$\begin{aligned} \pi \int_{a-t}^a y^2 dx &= \pi \int_{a-t}^a (a^2 - x^2) dx = \pi \left[ a^2 x - \frac{1}{3} x^3 \right]_{a-t}^a \\ &= \pi (at^2 - \frac{1}{3} t^3). \end{aligned}$$

As a quick check on the algebra, notice that this is zero when  $t = 0$  and  $\frac{4}{3}\pi a^3$  when  $t = 2a$ .

Thus  $V/A = (3at - t^2)/(6a)$ . This is a quadratic curve with zeros at  $t = 0$  and  $t = 3a$ , so it has a maximum at  $t = \frac{3}{2}a$  (by differentiating or otherwise), where  $V/A = \frac{3}{8}a$ . Since we require this ratio to be less than one, we must have  $a < \frac{8}{3}$  metres.

## Post-mortem

It was well worth studying the information given at the end of the question before plunging into it: it not only gave the necessary formulae for the surface area and volume, but also gave them in a form that suggested a way forward right at the start of the question.

Did you think about whether there are other shapes that would have the property proved for the sphere in the first paragraph of the question? Mathematically, it boils down to whether there are functions  $y(x)$ , other than our function  $y = \sqrt{a^2 - x^2}$ , that can satisfy (\*). If we differentiate (\*), we obtain

$$2\pi y \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)^{\frac{1}{2}} = 2\pi a$$

which looks formidable, but in fact simplifies to an equation that you can integrate quite easily. The sphere is, it turns out, the only shape with the required property.