## Problem 50: More curve sketching (✓✓✓)

**(i)** The curve $C_1$ passes through the origin in the $x$–$y$ plane and its gradient is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(1-x^2)\mathrm{e}^{-x^2}.$$

Show that $C_1$ has a minimum point at the origin and a maximum point at $\left(1, \frac{1}{2}\,\mathrm{e}^{-1}\right)$. Find the coordinates of the other stationary point. Give a rough sketch of $C_1$.

**(ii)** The curve $C_2$ passes through the origin and its gradient is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(1-x^2)\mathrm{e}^{-x^3}.$$

Show that $C_2$ has a minimum point at the origin and a maximum point at $(1, k)$, where $k > \frac{1}{2}\,\mathrm{e}^{-1}$. (You need not find $k$.)

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## Comments

No work is required to find the $x$ coordinate of the stationary points, but you have to integrate the differential equation to find the $y$ coordinate. For the second part, you cannot integrate the equation — other than numerically, or in terms of rather obscure special functions that you almost certainly haven't come across, such as the incomplete gamma function defined by

$$\Gamma(x, a) = \int_0^a t^{x-1}\mathrm{e}^{-t}\mathrm{d}t\,.$$

However, you can obtain an estimate, which is all that is required, by comparing the gradients of $C_1$ with $C_2$ and thinking of the graphs for $-1 \leqslant x \leqslant 1$. This is perhaps a bit tricky; an idea that you may well not alight on under examination conditions.

## Solution to problem 50

$C_1$ has stationary points when $\dfrac{dy}{dx} = 0$, i.e. when $x = 0$, $x = +1$ or $x = -1$. To find the $y$ coordinates of the stationary points, we integrate the differential equation, using integration by parts:

$$y = \int (1-x^2)xe^{-x^2}\,dx = (1-x^2)(-\tfrac{1}{2}e^{-x^2}) - \int xe^{-x^2}\,dx = -\tfrac{1}{2}(1-x^2)e^{-x^2} + \tfrac{1}{2}e^{-x^2} + \text{const} = \tfrac{1}{2}x^2e^{-x^2},$$

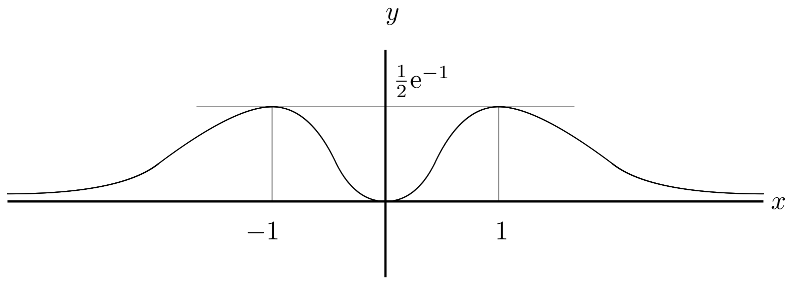
where we have used the fact that $C_1$ passes through the origin to evaluate the constant of integration. The coordinates of the stationary points are therefore $(1, \frac{1}{2}\,e^{-1})$, $(0,0)$ and $(-1, \frac{1}{2}\,e^{-1})$.

One way of classifying the stationary points is to look at the second derivative:

$$\frac{dy}{dx} = (x - x^3)e^{-x^2} \implies \frac{d^2y}{dx^2} = (1-3x^2)e^{-x^2} - 2x^2(1-x^2)e^{-x^2} = (1-5x^2+2x^4)e^{-x^2},$$

which is positive when $x = 0$ (indicating a minimum) and negative when $x = 1$ and $x = -1$ (indicating maxima).

Since $y \to 0$ as $x \to \pm\infty$, the sketch is:




For $C_2$, the stationary points are again at $x = 0$ and $x = \pm 1$. To classify the stationary points, we calculate the second derivative:

$$\frac{d^2y}{dx^2} = (1 - 3x^2 - 3x^3 + 3x^5)e^{-x^3},$$

which is positive at $x = 0$ (a minimum) and negative at $x = 1$ (a maximum).

Since we cannot integrate this differential equation explicitly, we must compare $C_2$ with $C_1$ to see whether the value of $y$ at the maxima is indeed greater for $C_2$ than for $C_1$.

For $0 < x < 1$, $x^3 < x^2$, so $e^{x^3} < e^{x^2}$ and $e^{-x^3} > e^{-x^2}$. Thus the gradient of $C_2$ is greater than the gradient of $C_1$. Since both curves pass through the origin, we deduce that $C_2$ lies above $C_1$ for $0 < x \leqslant 1$ and therefore the maximum point $(1, k)$ on $C_2$ has $k > \frac{1}{2}e^{-1}$.

## Post-mortem

You were probably surprised that the examiners didn't ask you to sketch $C_2$; they usually do. It looks as if it could be done, because you know that $C_2$ passes through the origin and you can relate its slope, roughly at least, to that of $C_1$, which you did sketch. The difficulty lies in determining its behaviour as $x \to \infty$; does $y \to 0$ as for $C_1$? In fact, it doesn't; no reason why it should. Instead, $y \to 0.153\ldots$ . We shouldn't have expected $C_1$ to asymptote to the $x$-axis either; the fact that it does so is due to a rather delicate balancing of the two terms that determine its gradient.