

Problem 48: Solving a quartic

(✓✓)

Given that

$$x^4 + px^2 + qx + r \equiv (x^2 - ax + b)(x^2 + ax + c),$$

express p , q and r in terms of a , b and c .

Show that a^2 is a root of the cubic equation

$$u^3 + 2pu^2 + (p^2 - 4r)u - q^2 = 0.$$

Verify that $u = 9$ is a root in the case $p = -1$, $q = -6$, $r = 15$.

Hence, or otherwise, solve the equation

$$y^4 - 8y^3 + 23y^2 - 34y + 39 = 0.$$

2000 Paper III

Comments

The long-sought solution of the general cubic was found, in 1535, by Niccolò Tartaglia (c. 1500–1557). He was persuaded to divulge his secret (in the form of a poem) by Girolamo Cardano (1501–1576), who promised not to publish it before Tartaglia did. However, Cardano discovered that it had previously been discovered by del Ferro (1465–1525/6) before 1515 so he published it himself in his algebra book *The Great Art*. There followed an acrimonious dispute between Tartaglia and Cardano, in which the latter was championed by his student Ferrari (1522–1565). The dispute culminated in a public mathematical duel between Ferrari and Tartaglia held in the church of Santa Maria in Milan in 1548, in which they attempted to solve each others' cubics. The duel ended in a shouting match with Tartaglia storming off. It seems Ferrari was the winner. Tartaglia was sacked from his job as lecturer and Ferrari made his fortune as a tax assessor before becoming a professor of mathematics at Bologna. He was poisoned by his sister, with arsenic, in 1565.

Ferrari found a way of reducing quartic equations to cubic equations; his method (roughly) is used in this question to solve a quartic which could probably be solved easier 'otherwise'. But it is the method that is interesting, not the solution.

The first step of the Ferrari method is to reduce the general quartic to a quartic equation with the cubic term missing by means of a linear transformation of the form $x \rightarrow x - a$. Then this reduced quartic is factorised (the first displayed equation in this question). The factorisation can be found by solving a cubic equation (the second displayed equation above) that must be satisfied by one of the coefficients in the factorised form.

The solutions of the quartic are all complex, but don't worry if you haven't come across complex numbers: you will be able to do everything except perhaps write down the last line.

You will no doubt be full of admiration for this clever method of solving quartic equations. One thing you are bound to ask yourself is how the other two roots of the cubic equation fit into the picture. In fact, the cubic equation gives (in general) 6 distinct values of a so there is quite a lot of explaining to do, given that the quartic has at most 4 distinct roots.

Solution to problem 48

We have

$$\begin{aligned}(x^2 - ax + b)(x^2 + ax + c) &= (x^2 - ax)(x^2 + ax) + b(x^2 + ax) + c(x^2 - ax) + bc \\ &= x^4 + (b + c - a^2)x^2 + a(b - c)x + bc\end{aligned}$$

so

$$p = b + c - a^2, \quad q = a(b - c), \quad r = bc. \quad (*)$$

To obtain an equation for a in terms of p , q and r we eliminate b and c from $(*)$ using the identity $(b + c)^2 = (b - c)^2 + 4bc$. This gives $(p + a^2)^2 = (q/a)^2 + 4r$ which simplifies easily to the given cubic with u replaced by a^2 .

We can easily verify by direct substitution that $u = 9$ satisfies the given cubic.

To solve the quartic equation, the first task is to reduce it to the form $x^4 + px^2 + qx + r = 0$, which has no term in x^3 . This is done by means of a translation. Noting that $(y - a)^4 = y^4 - 4ay^3 + \dots$, we set $x = y - 2$. This gives

$$(x + 2)^4 - 8(x + 2)^3 + 23(x + 2)^2 - 34(x + 2) + 39 = 0.$$

which (not surprisingly) boils down to $x^4 - x^2 - 6x + 15 = 0$, so that $p = -1$, $q = -6$ and $r = 15$. We have already shown that one root of the cubic corresponding to these values is $u = 9$. Thus we can achieve the factorisation of the quartic into two quadratic factors by setting $a = 3$ (or $a = -3$; it doesn't matter which we use) and

$$\begin{aligned}b &= \frac{1}{2} \left(p + a^2 + \frac{q}{a} \right) = 3 \\ c &= \frac{1}{2} \left(p + a^2 - \frac{q}{a} \right) = 5.\end{aligned}$$

Thus

$$x^4 - x^2 - 6x + 15 = (x^2 - 3x + 3)(x^2 + 3x + 5).$$

Setting each of the two quadratic factors equal to zero gives

$$x = \frac{3 \pm i\sqrt{3}}{2} \quad \text{and} \quad x = \frac{-3 \pm i\sqrt{11}}{2},$$

so

$$y = \frac{7 \pm i\sqrt{3}}{2} \quad \text{or} \quad y = \frac{1 \pm i\sqrt{11}}{2}.$$

Post-mortem

Did you work out the relation between the six possible values of a (corresponding to given values of p , q and r) and the roots of the quartic? The point is that the quartic can be written as the product of four linear factors (by the fundamental theorem of algebra) and there are six ways of grouping the linear factors into two quadratic factors. Each way corresponds to a value of a .