

Problem 47: Vector geometry

(✓✓)

The line ℓ has vector equation $\mathbf{r} = \lambda \mathbf{s}$, where

$$\mathbf{s} = (\cos \theta + \sqrt{3}) \mathbf{i} + \sqrt{2} \sin \theta \mathbf{j} + (\cos \theta - \sqrt{3}) \mathbf{k}$$

and λ is a scalar parameter. Find an expression for the angle between ℓ and the line $\mathbf{r} = \mu(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$. Show that there is a line m through the origin such that, whatever the value of θ , the acute angle between ℓ and m is $\frac{1}{6}\pi$.

A plane has equation $x - z = 4\sqrt{3}$. The line ℓ meets this plane at P . Show that, as θ varies, P describes a circle, with its centre on m . Find the radius of this circle.

2000 Paper II

Comments

It is not easy to set vector questions at this level: they tend to become merely complicated and tedious, rather than difficult in an interesting way. In a good question, there is usually some underlying geometry and it pays to try to understand what this is. Here, the question is about the geometrical object traced out by ℓ as θ varies.

You will need to know about scalar products of vectors for this question, but otherwise it is really just coordinate geometry.

Vectors form an extremely important part of almost every branch of mathematics (maybe every branch of mathematics) and will probably be one of the first topics you tackle on your university course.

Solution to problem 47

Both ℓ and the line $\mathbf{r} = \mu(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$ pass through the origin, so the angle α between the lines is given by the scalar product of the unit vectors, i.e. the scalar product between the given vectors divided by the product of the lengths of the two vectors:

$$\begin{aligned}\cos \alpha &= \frac{a(\cos \theta + \sqrt{3}) + b(\sqrt{2} \sin \theta) + c(\cos \theta - \sqrt{3})}{\sqrt{(\cos \theta + \sqrt{3})^2 + 2 \sin^2 \theta + (\cos \theta - \sqrt{3})^2} \sqrt{a^2 + b^2 + c^2}} \\ &= \frac{(a + c) \cos \theta + \sqrt{2} b \sin \theta + (a - c) \sqrt{3}}{2\sqrt{2} \sqrt{a^2 + b^2 + c^2}}.\end{aligned}\quad (*)$$

Now we want to show that there is some choice of a , b and c such that $\cos \alpha$ does not depend on the value of θ . By inspection of equation (*), we see that this requires $a = -c$ and $b = 0$.

Setting $a = -c$ in (*) gives $\cos \alpha = \frac{1}{2}\sqrt{3}$ and $\alpha = \frac{1}{6}\pi$ as required. We can absorb the constant a into μ , so the equation of the line m becomes

$$\mathbf{r} = \mu(\mathbf{i} - \mathbf{k}). \quad (**)$$

The coordinates of a general point on the line ℓ are

$$x = \lambda(\cos \theta + \sqrt{3}), \quad y = \lambda\sqrt{2} \sin \theta, \quad z = \lambda(\cos \theta - \sqrt{3}).$$

For a point which is also on the plane $x - z = 4\sqrt{3}$ we have

$$\lambda(\cos \theta + \sqrt{3}) - \lambda(\cos \theta - \sqrt{3}) = 4\sqrt{3}$$

so $\lambda = 2$. The point P at the intersection between the line and the plane therefore has coordinates

$$(2 \cos \theta + 2\sqrt{3}, 2\sqrt{2} \sin \theta, 2 \cos \theta - 2\sqrt{3}). \quad (***)$$

As θ varies, does P move round a circle? That is not easy to see, but fortunately we gather from the question that the centre of the circle is on m . It must also lie in the plane of the circle, which is the plane $x - z = 4\sqrt{3}$. The line (**) meets this plane at the point $(2\sqrt{3}, 0, -2\sqrt{3})$; call it O . To verify that P describes a circle with centre O we must check that the distance from P , given by (***), to O is independent of θ . We have

$$OP^2 = (2 \cos \theta)^2 + (2\sqrt{2} \sin \theta)^2 + (2 \cos \theta)^2 = 8,$$

which is indeed independent of θ . The radius of the circle is therefore $2\sqrt{2}$.

Post-mortem

As mentioned in the comments section, it is helpful to understand the geometry of vector questions. Since the angle between the variable line ℓ and the fixed line m is constant ($\frac{1}{6}\pi$), the shape generated by ℓ as θ varies is the surface of a cone (actually a pair of cones).

The intersection of a plane with a cone is in general a *conic section*: an ellipse (of which a circle is a special case), hyperbola, parabola or pair of straight lines. Try to picture these possibilities. In this case, the normal to the plane, which is in direction $(1, 0, -1)$, is parallel to the axis of the cone (the line m), so the intersection is indeed a circle.