

Problem 46: Bernoulli polynomials

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The Bernoulli polynomials, $B_n(x)$ (where $n = 0, 1, 2, \dots$), are defined by $B_0(x) = 1$ and, for $n \geq 1$,

$$\frac{dB_n}{dx} = nB_{n-1}(x) \quad (1)$$

and

$$\int_0^1 B_n(x) dx = 0. \quad (2)$$

(i) Show that $B_4(x) = x^2(x-1)^2 + c$, where c is a constant (which you need not evaluate).

(ii) Show that, for $n \geq 2$, $B_n(1) - B_n(0) = 0$.

(iii) Show, by induction or otherwise, that

$$B_n(x+1) - B_n(x) = nx^{n-1} \quad (n \geq 1). \quad (3)$$

(iv) Hence show that

$$n \sum_{m=0}^k m^{n-1} = B_n(k+1) - B_n(0),$$

and deduce that $\sum_{m=0}^{1000} m^3 = (500500)^2$.

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Comments

The Swiss family Bernoulli included no fewer than eight mathematicians who were counted amongst the leading scholars of their day. They made major contributions to all branches of mathematics, especially differential calculus. There was great rivalry between some members of the family; between brothers Jakob (1654–1705) and Johann (1667–1748), in particular.

Johann once published an important result in the form of a Latin anagram, in order to retain the priority of discovery without giving the game away to his brother. The anagram was: *24a, 6b, 6c, 8d, 33e, 5f, 2g, 4h, 33i, 6l, 21m, 26n, 16o, 8p, 5q, 17r, 16s, 25t, 32u, 4x, 3y, +, -, -, ±, =, 4, 2, 1, '.* The notation means that his important result contained, for instance, the letter *a* 24 times either in text or in equations. After waiting for a year for someone to solve it, Bernoulli weakened and published the solution himself. If you are trying to solve the anagram yourself, you might like to know that it is about the Riccati equation $y' = ay^2 + bx^n$, which can be solved (very cunningly, as it turns out) when n is of the form $-4m/(2m \pm 1)$ for any positive integer m . (Newton also published some work in the form of anagrams, during his conflict with Leibniz.)

The polynomials described above were discovered by Jakob Bernoulli. They are defined recursively; that is to say, the zeroth polynomial is given an explicit value, and the n th is determined from the $(n-1)$ th. Here, B_{n-1} has to be integrated to obtain B_n , which means that B_n is a polynomial of degree n . The constant of integration is determined by the condition (2), so B_n is uniquely determined. We have to do this explicitly for part (i).

Solution to problem 46

(i) First we find $B_1(x)$ by integrating $1 \times B_0(x)$, using equation (1): $B_1(x) = x + k$, where k is a constant. We find k by applying the condition $\int_0^1 B_1(x) dx = 0$, which gives $k = -\frac{1}{2}$. Next we find $B_2(x)$ by similar means, giving $x^2 - x + \frac{1}{6}$, and similarly $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ and $B_4(x)$ is as given.

(ii) We are asked to prove a result involving $B_n(x)$ evaluated at $x = 1$ and $x = 0$, i.e. at the limits of the integral (2). We therefore try the effect of integrating both sides of equation (1) between these limits:

$$B_n(1) - B_n(0) \equiv \int_0^1 \frac{dB_n(x)}{dx} dx = n \int_0^1 B_{n-1}(x) dx = 0,$$

using property (2) with n replaced by $(n - 1)$.

(iii) First the easy bit of the induction proof. For $n = 1$, we have $B_1(x) = x - \frac{1}{2}$, so

$$B_1(x+1) - B_1(x) = (x+1 - \frac{1}{2}) - (x - \frac{1}{2}) = 1 \equiv nx^{n-1},$$

so the formula holds.

Now suppose that it holds for $n = k$:

$$B_k(x+1) - B_k(x) - kx^{k-1} = 0 \tag{4}$$

and investigate

$$B_{k+1}(x+1) - B_{k+1}(x) - (k+1)x^k, \tag{5}$$

which we hope will also equal zero.

The only helpful thing we know about Bernoulli polynomials involves the derivatives. Therefore, let us see what happens when we differentiate the expression (5):

$$\frac{d}{dx} B_{k+1}(x+1) - \frac{d}{dx} B_{k+1}(x) - (k+1)kx^{k-1}.$$

Now using (1) gives

$$(k+1)B_k(x+1) - (k+1)B_k(x) - (k+1)kx^{k-1}.$$

Note that we have used the chain rule to differentiate $B_{k+1}(x+1)$ with respect to x rather than with respect to $(x+1)$. Note also that there is a pleasing overall factor of $(k+1)$, which suggests that we are on the right track. In fact, taking out this factor gives exactly the left hand side of equation (4), which is zero.

Of course, we are not finished yet: we have only shown that the derivative of equation (5) is equal to zero; the expression (5) is therefore constant:

$$B_{k+1}(x+1) - B_{k+1}(x) - (k+1)x^k = A.$$

We must show that $A = 0$. Setting $x = 0$ gives $B_{k+1}(1) - B_{k+1}(0) = A$, which implies that $A = 0$ by part (ii).

(iv) Summing (3) from $x = 0$ to $x = k$ gives the first of these results immediately because nearly all the terms cancel in pairs. The evaluation of the sum follows by calculating $B_4(1001) - B_4(0)$ from the result given in part (i).