

Problem 45: Lagrange's identity

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If $y = f(x)$, the inverse of f is given by *Lagrange's identity*:

$$f^{-1}(y) = y + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^{n-1}}{dy^{n-1}} [y - f(y)]^n,$$

when this series converges.

(i) Verify Lagrange's identity when $f(x) = ax$.

(ii) Show that one root of the equation $x - \frac{1}{4}x^3 = \frac{3}{4}$ is

$$x = \sum_{n=0}^{\infty} \frac{3^{2n+1} (3n)!}{n! (2n+1)! 4^{3n+1}}. \quad (\dagger)$$

(iii) Find a solution for x , as a series in λ , of the equation $x = e^{\lambda x}$.

[You may assume that the series in part (ii) converges and that the series in parts (i) and (iii) converge for suitable a and λ .]

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Comments

This looks pretty frightening at first, because of the complicated and unfamiliar formula. However, its bark is worse than its bite. Once you have decided what you need to find the inverse of, you just substitute it into the formula and see what happens. Do not worry about the use of the word 'convergence'; this can be ignored. It is just included to satisfy the legal eagles who will point out that the series might not have a finite sum.

In part (ii) you can, as it happens, solve the cubic by normal means (find one root by inspection, factorise and use the usual formula to solve the resulting quadratic equation). The root found by Lagrange's equation is the one closest to zero. Equation ($\dagger$) turns out to be a very obscure way of writing a familiar quantity.²⁸

Lagrange was one of the leading mathematicians of the 18th century; Napoleon referred to him as the 'lofty pyramid of the mathematical sciences'. He attacked a wide range of problems, from celestial mechanics to number theory. In the course of his investigation of the roots of polynomial equations, he discovered group theory (in particular, his eponymous theorem about the order of a subgroup dividing the order of the group), though the term 'group' and the systematic theory had to wait until Galois and Abel in the first part of the 19th century.

Lagrange's formula, produced before the advent of the theory of integration in the complex plane, which allows a relatively straightforward derivation, testifies to his remarkable mathematical ability. It is practically forgotten now, but in its day it had a great impact. The applications given above give an idea of how important it was, in the age before computers.

²⁸ The expansion sums to 1; I don't know how you can see that directly. I thought it would come from using the classical formula for the root of a cubic:

$$-\left[\frac{3}{2}\left(1 + i\sqrt{\frac{13}{243}}\right)\right]^{\frac{1}{3}} - \left[\frac{3}{2}\left(1 - i\sqrt{\frac{13}{243}}\right)\right]^{\frac{1}{3}},$$

expanding each bracket binomially but it doesn't seem to. The roots are obtained from this complicated expression by noticing that $-\left[\frac{3}{2}\left(1 \pm i\sqrt{\frac{13}{243}}\right)\right]^{\frac{1}{3}} = \frac{1}{2}\left(1 \pm i\sqrt{\frac{13}{3}}\right)$.

Solution to problem 45

(i) The inverse of $f(x) = ax$ is given by $f^{-1}(y) = y/a$. Substituting into $f(y) = ay$ into Lagrange's identity gives

$$\begin{aligned} f^{-1}(y) &= y + \sum_1^{\infty} \frac{1}{n!} \frac{d^{n-1}}{dy^{n-1}} [y - ay]^n = y + \sum_1^{\infty} (1-a)^n \frac{1}{n!} \frac{d^{n-1} y^n}{dy^{n-1}} \\ &= y + \sum_1^{\infty} (1-a)^n y = y + y \frac{1-a}{1-(1-a)}, \end{aligned}$$

where the last equality follows from summing the geometric progression. This simplifies to y/a , thus verifying Lagrange's formula.

(ii) Let $f(x) = x - \frac{1}{4}x^3$. Then the equation becomes $f(x) = \frac{3}{4}$, so we must find $f^{-1}(\frac{3}{4})$. Again, we just substitute into Lagrange's formula, leaving y arbitrary for the moment:

$$\begin{aligned} f^{-1}(y) &= y + \sum_1^{\infty} \frac{1}{n!} \frac{d^{n-1}}{dy^{n-1}} [y - (y - \frac{1}{4}y^3)]^n = y + \sum_1^{\infty} \frac{1}{n!} \frac{d^{n-1}}{dy^{n-1}} [\frac{1}{4}y^3]^n \\ &= y + \sum_1^{\infty} \frac{1}{4^n n!} \frac{d^{n-1}}{dy^{n-1}} y^{3n} \\ &= y + \sum_1^{\infty} \frac{1}{4^n n!} \frac{(3n)!}{(2n+1)!} y^{2n+1}. \end{aligned}$$

This is a solution to the equation²⁹ $x - \frac{1}{4}x^3 = y$, so we just set $y = \frac{3}{4}$ to obtain the given result.

(iii) The obvious choice for f is $f(x) = x - e^{\lambda x}$, in which case the equation becomes $f(x) = 0$ and we want $f^{-1}(0)$. Again substituting into Lagrange's identity gives

$$f^{-1}(y) = y + \sum_1^{\infty} \frac{1}{n!} \frac{d^{n-1}}{dy^{n-1}} [e^{\lambda y}]^n = y + \sum_1^{\infty} \frac{1}{n!} \frac{d^{n-1}}{dy^{n-1}} e^{ny\lambda} = y + \sum_1^{\infty} \frac{1}{n!} (n\lambda)^{n-1} e^{ny\lambda}.$$

Setting $y = 0$ gives a series for the root:

$$f^{-1}(0) = \sum_1^{\infty} \frac{n^{n-1}}{n!} \lambda^{n-1},$$

which cannot be further simplified.

Post-mortem

Regarding convergence in Lagrange's formula:

Part (i): we need, somewhat mysteriously, $0 < a < 2$ for the geometric progression to converge, but then the result is valid for any y .

Part (ii): Using the approximation $n! \approx (n/e)^n$, which is a simplified version of Stirling's formula, it can be seen that the series converges provided $|y| < 4/\sqrt{27}$. This is related to the condition for x to lie between the two turning points of $f(x)$, which guarantees that $f^{-1}(y)$ is well defined.

Part (iii): We can use Stirling's formula, as above, to show that series converges for $|\lambda| < e^{-1}$. You might like to sketch the two functions x and $e^{\lambda x}$; you should find that the range of values of λ for which the equation has a real root corresponds exactly to the range for which the series converges.

²⁹ Note that this equation cannot in general be solved by spotting roots. In fact, by translation and scaling, any cubic equation can be reduced to this form, so our series solution can be used to find a solution of any cubic equation.