

Problem 43: The t substitution

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Show that

$$\sin \theta = \frac{2t}{1+t^2}, \quad \cos \theta = \frac{1-t^2}{1+t^2}, \quad \text{and} \quad \frac{1+\cos \theta}{\sin \theta} = \tan\left(\frac{1}{2}\pi - \frac{1}{2}\theta\right),$$

where $t = \tan(\frac{1}{2}\theta)$.

Let

$$I = \int_0^{\frac{1}{2}\pi} \frac{1}{1 + \cos \alpha \sin \theta} d\theta.$$

Use the substitution $t = \tan(\frac{1}{2}\theta)$ to show that, for $0 < \alpha < \frac{1}{2}\pi$,

$$I = \int_0^1 \frac{2}{(t + \cos \alpha)^2 + \sin^2 \alpha} dt.$$

By means of the further substitution $t + \cos \alpha = \sin \alpha \tan u$ show that

$$I = \frac{\alpha}{\sin \alpha}.$$

Deduce a similar result for

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1 + \sin \alpha \cos \phi} d\phi.$$

2000 Paper II

Comments

The first of the two substitutions is familiarly known as the ' t substitution'. It would have been very standard fare 30 years ago, but it seems to have gone out of fashion now. The second of the two substitutions is the normal substitution for integrals with quadratic denominators.

For the last part, 'deduce' implies that you don't have to do any further integration. Note that the variable in the integral is ϕ instead of θ . Since it is a definite integral, it doesn't matter what the variable is called; it could equally well have been called θ as in the original integral I . The use of a different variable was just a kindness on the part of the examiner to indicate that you should be thinking about a change of variable.

The 'similar result' that you deduce should include the conditions under which it is true. It is worth thinking about why the condition $0 < \alpha < \frac{1}{2}\pi$ is required — or, indeed, if it is required.

Solution to problem 43

The three identities just require use of $\cos \theta = \cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta$ and $\sin \theta = 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta$. If you divide each by $\cos^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\theta$ (i.e. by 1) the first two identities drop out. Remember, for the last one, that $\cot x = \tan(\frac{1}{2}\pi - x)$.

For the first change of variable, we have $dt = \frac{1}{2} \sec^2 \frac{1}{2}\theta d\theta = \frac{1}{2}(1+t^2)d\theta$ and the new limits are 0 and 1, so

$$\begin{aligned} I = \int_0^{\frac{1}{2}\pi} \frac{1}{1 + \cos \alpha \sin \theta} d\theta &= \int_0^1 \frac{1}{1 + \cos \alpha \frac{2t}{1+t^2}} \frac{2}{1+t^2} dt \\ &= \int_0^1 \frac{2}{1 + 2t \cos \alpha + t^2} dt = \int_0^1 \frac{2}{(t + \cos \alpha)^2 + \sin^2 \alpha} dt. \end{aligned}$$

For the second change of variable, we have $dt = \sin \alpha \sec^2 u du$. When $t = 0$, $\tan u = \cot \alpha$ so $u = \frac{1}{2}\pi - \alpha$. When $t = 1$, $\sin \alpha \tan u = 1 + \cos \alpha$ so (after a bit of work with double-angle formulae) $u = \frac{1}{2}\pi - \frac{1}{2}\alpha$. Thus

$$I = \int_{\frac{1}{2}\pi - \alpha}^{\frac{1}{2}\pi - \frac{1}{2}\alpha} \frac{2}{\sin^2 \alpha (1 + \tan^2 u)} \sin \alpha \sec^2 u du = \int_{\frac{1}{2}\pi - \alpha}^{\frac{1}{2}\pi - \frac{1}{2}\alpha} \frac{2}{\sin \alpha} du = \frac{\alpha}{\sin \alpha}.$$

For the last part, we want to make a substitution that changes the cosine in the denominator to a sine. One possibility is to set $\phi = \frac{1}{2}\pi - \theta$. This will swap the limits but also introduces a minus sign since $d\theta = -d\phi$. Thus

$$\frac{\alpha}{\sin \alpha} = I = - \int_{\frac{\pi}{2}}^0 \frac{1}{1 + \cos \alpha \cos \phi} d\phi = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos \alpha \cos \phi} d\phi.$$

This is almost the integral we want: we still need to replace $\cos \alpha$ in the denominator by $\sin \alpha$. Remembering what we did a couple of lines back, we just replace α by $\frac{1}{2}\pi - \alpha$ in the integral and in the answer, giving

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos \alpha \cos \phi} d\phi = \frac{\frac{1}{2}\pi - \alpha}{\cos \alpha}.$$

If the original α satisfied $0 < \alpha < \frac{1}{2}\pi$, the new α must satisfy $\frac{1}{2}\pi > \alpha > 0$, which is the same.

Post-mortem

One reason for the restriction on α might be to prevent the denominator of the integrand being zero for some value of θ ; a zero in the denominator usually means that the integral is undefined. However, the only value of α for which $\cos \alpha \sin \theta$ could possibly be as small as -1 (for $0 \leq \theta \leq \frac{1}{2}\pi$) is π (and of course 3π , etc). From this point of view, we only need $\alpha \neq \pi$ (etc). There is a slight awkwardness in the answer when $\alpha = 0$ but this can be overcome by taking limits:

$$\lim_{\alpha \rightarrow 0} \frac{\alpha}{\sin \alpha} = 1$$

which you can easily verify is the correct value of the integral when $\alpha = 0$.

You might therefore think that the restriction $0 < \alpha < \frac{1}{2}\pi$ is superfluous. But here is a curious thing: increasing α by 2π does not change I but does change the answer! If you work through the solution with this in mind you see that it can make a difference only when you work out $\tan^{-1} u$, which by definition lies in the range $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, and this is why we must have $-\frac{1}{2}\pi < \alpha \leq \frac{1}{2}\pi$. We lose nothing by using instead $0 \leq \alpha \leq \frac{1}{2}\pi$, since $\alpha \rightarrow -\alpha$ doesn't change the integral. The strict inequalities ($<$ rather than $\leq$) avoid trouble with the denominators when $\alpha = 0$ in the first integral or $\alpha = \frac{1}{2}\pi$ in the second.