Problem 40: Estimating the value of an integral (✓✓)

(i) Show that, for $0 \leqslant x \leqslant 1$, the largest value of $\dfrac{x^6}{(x^2+1)^4}$ is $\dfrac{1}{16}$.

What is the smallest value?

(ii) Find constants A, B, C and D such that, for all x,

$$\frac{1}{(x^2+1)^4} \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Ax^5+Bx^3+Cx}{(x^2+1)^3}\right) + \frac{Dx^6}{(x^2+1)^4}\,.$$

(iii) Hence, or otherwise, prove that

$$\frac{11}{24} \leqslant \int_0^1 \frac{1}{(x^2+1)^4}\,\mathrm{d}x \leqslant \frac{11}{24} + \frac{1}{16}\,.$$

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Comments

You should think about part (i) graphically, though it is not necessary to draw the graph: just set about it as if you were going to (starting point, finishing point, turning points, etc).

The equivalence sign in part (ii) indicates an equality that holds for all x — you are not being asked to solve the equation for x.

For part (iii), you need to know that inequalities can be integrated: this is 'obvious' if you think about integration in terms of area, though a formal proof requires a formal definition of integration and this is the sort of thing you would do in your first university course in mathematical analysis.

Solution to problem 40

(i) Let $\mathrm{f}(x) = \dfrac{x^6}{(x^2+1)^4}$. Then

$$\mathrm{f}'(x) = \frac{6x^5}{(x^2+1)^4} - \frac{8x^7}{(x^2+1)^5} = \frac{2(3-x^2)x^5}{(x^2+1)^5}$$

which is positive for $0 < x^2 < 3$. Therefore, $\mathrm{f}(x)$ increases in value from 0 at $x = 0$ to $\frac{1}{16}$ at $x = 1$.

(ii) Doing the differentiation gives

$$\frac{1}{(x^2+1)^4} \equiv \frac{5Ax^4 + 3Bx^2 + C}{(x^2+1)^3} - \frac{6x(Ax^5 + Bx^3 + Cx)}{(x^2+1)^4} + \frac{Dx^6}{(x^2+1)^4}$$

and multiplying by $(x^2+1)^4$ gives the following identity:

$$1 \equiv (5Ax^4 + 3Bx^2 + C)(x^2+1) - 6x(Ax^5 + Bx^3 + Cx) + Dx^6 \tag{$*$}$$

i.e.

$$1 \equiv (D - A)x^6 + (5A - 3B)x^4 + (3B - 5C)x^2 + C\ .$$

Equating coefficients of the different powers of x on each side of the equivalence sign gives $1 = C$, $0 = 3B - 5C$, $0 = 5A - 3B$, $0 = D - A$, so $A = 1$, $B = \frac{5}{3}$, $C = 1$ and $D = 1$.

(iii) Using the results of parts (i) and (ii), we see that for $0 \leqslant x \leqslant 1$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^5 + \frac{5}{3}x^3 + x}{(x^2+1)^3}\right) \leqslant \frac{1}{(x^2+1)^4} \leqslant \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^5 + \frac{5}{3}x^3 + x}{(x^2+1)^3}\right) + \frac{1}{16}\ .$$

Inequalities can be integrated, so

$$\int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^5 + \frac{5}{3}x^3 + x}{(x^2+1)^3}\right)\mathrm{d}x \leqslant \int_0^1 \frac{1}{(x^2+1)^4}\,\mathrm{d}x \leqslant \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^5 + \frac{5}{3}x^3 + x}{(x^2+1)^3}\right)\mathrm{d}x + \int_0^1 \frac{1}{16}\,\mathrm{d}x$$

i.e.

$$\left[\frac{x^5 + (5/3)x^3 + x}{(x^2+1)^3}\right]_0^1 \leqslant \int_0^1 \frac{1}{(x^2+1)^4}\,\mathrm{d}x \leqslant \left[\frac{x^5 + (5/3)x^3 + x}{(x^2+1)^3}\right]_0^1 + \left[\frac{x}{16}\right]_0^1$$

from which the required result follows immediately.

Post-mortem

Instead of equating coefficients in ($*$), you could obtain equations for A, B, C and D by putting four carefully chosen values of x into the equation. An obvious choice is $x = 0$, but (thinking flexibly!) you could try $x = \mathrm{i}$ to eliminate terms with factors of $x^2 + 1$. After this, it becomes more difficult to find good choices.

You might wonder why, in part (ii), the term inside the derivative has only odd powers of x. Would it not make it more general to include even powers as well? You could in fact include even powers but you would find that their coefficients would be zero: all the other terms in the equation are even in x, so the derivative has to be an even function which means that the function being differentiated must be odd.

Although the idea of this question is good, the final result is a bit feeble. It only gives the value of the integral to an accuracy of about $\frac{1}{16}/\frac{11}{24}$ which is about 15%. The actual value of the integral can be found fairly easily using the substitution $x = \tan t$ and is $\frac{11}{48} + \frac{5}{64}\pi$ so the inequalities can be used to give (rather bad) estimates for π, namely $2.933 \leqslant \pi \leqslant 3.733$.