Problem 39: A difficult integral (✓✓✓)

Given that $\tan \frac{1}{4}\pi = 1$ show that $\tan \frac{1}{8}\pi = \sqrt{2} - 1$.
Let

$$I = \int_{-1}^{1} \frac{1}{\sqrt{1+x} + \sqrt{1-x} + 2} \, \mathrm{d}x \, .$$

Show, by using the change of variable $x = \sin 4t$, that

$$I = \int_{0}^{\frac{1}{8}\pi} \frac{2\cos 4t}{\cos^2 t} \, \mathrm{d}t \, .$$

Hence show that

$$I = 4\sqrt{2} - \pi - 2 \, .$$

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Comments

This tests trigonometric manipulation and integration skills. You will certainly need $\tan 2\theta$ in terms of $\tan\theta$, and $\cos 2\theta$ in terms of $\cos\theta$, and maybe other formulae.

Both parts of the question are *multistep*: there are half a dozen consecutive steps, each different in nature, with no guidance. This is unusual in school-level mathematics but normal in university mathematics.

I checked the answer on Wolfram Alpha, which turned out to be very good indeed at doing this sort of thing. I asked it to do the indefinite integral as well and, in less than a second, it came up with

$$\sqrt{1-x}\,\left[-1 - (\sqrt{x+1}+1)^{-1}\right] + [\sqrt{x+1}+1]^{-1} - 2\sin^{-1}\sqrt{\tfrac{1}{2}(x+1)}\, .$$

Not a pretty sight and not in its neatest form by a long way: for example, $2\sin^{-1}\sqrt{\frac{1}{2}(x+1)}$ reduces, after a bit of algebra, to $\frac{1}{2}\pi + \sin^{-1} x$. I also asked it to do the same integral with the 2 replaced by a parameter k and it took four seconds. The answer was about 20 times longer than the $k = 2$ result but it seemed to enjoy the problem, as far as I could tell.

Solution to problem 39

For the first part, to save writing, let $t = \tan\frac{\pi}{8}$. Then

$$\frac{2t}{1-t^2} = 1 \Rightarrow t^2 + 2t - 1 = 0 \Rightarrow t = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}\,.$$

We take the root with the $+$ sign since we know that t is positive.

Now the integral. Note first that the integrand has an obvious symmetry: it is unchanged when $x \leftrightarrow -x$. This means that we can do the integral over the half-range $x = 0$ to $x = 1$ and double the result. A glance at the required result suggests that this is a good idea.

Substituting $x = \sin 4t$ as instructed then gives

$$I = 2\int_0^{\frac{1}{8}\pi} \frac{4\cos 4t}{\sqrt{1+\sin 4t} + \sqrt{1-\sin 4t} + 2}\,\mathrm{d}t$$

so to obtain the given answer we need to show that

$$\sqrt{1+\sin 4t} + \sqrt{1-\sin 4t} + 2 = 4\cos^2 t\,,$$

i.e.

$$\sqrt{1+\sin 4t} + \sqrt{1-\sin 4t} = 2\cos 2t\,.$$

If we square both sides of this equation, noting that both sides are positive for the values of t in the integral so this is not dangerous, nice things happen:

$$(1+\sin 4t) + (1-\sin 4t) + 2\sqrt{1-\sin^2 4t} = 4\cos^2 2t \qquad \text{(RTP)}$$

i.e.

$$2 + 2\cos 4t = 4\cos^2 2t \qquad \text{(RTP)}$$

which is true by a standard trig. identity, so we have proved what we were required to prove.

For the last part, we have

$$\cos 4t = 2\cos^2 2t - 1 = 2(2\cos^2 t - 1)^2 - 1 = 8\cos^4 t - 8\cos^2 t + 1$$

so

$$I = 2\int_0^{\frac{1}{8}\pi} \big(8\cos^2 t - 8 + \sec^2 t\big)\,\mathrm{d}t = 2\int_0^{\frac{1}{8}\pi} \big(4\cos 2t - 4 + \sec^2 t\big)\,\mathrm{d}t = 2\Big[2\sin 2t - 4t + \tan t\Big]_0^{\frac{1}{8}\pi},$$

which gives the required result.

Post-mortem

Manipulating the integrand after the change of variable was really quite demanding. You could easily go down the wrong track and become mired in algebra. I did it by writing down what I was trying to prove and then showing that it was indeed true. This of course is hazardous, because if you are not careful you might assume the result in order to prove the result. I find it helps to write 'RTP' (Required To Prove) in the margin to indicate clearly to myself and others that I am not assuming it to be true.

The first thing we did with the integral, guided by the given answer, was to use the symmetry $x \leftrightarrow -x$ to reduce the range of integration to $0 \leqslant x \leqslant 1$, doubling the result. It is clear from a graph that this works but you could, if you were unsure, split the integral into two parts (integral from -1 to 0 plus integral from 0 to 1) then make the change of variable $y = -x$ in the lower integral to show that the two parts are equal.