Problem 38: Discontinuous integrands (✓✓)

For any number x, the largest integer less than or equal to x is denoted by $[x]$. For example, $[3.7] = 3$ and $[4] = 4$.

(i) Sketch the graph of $y = [x]$ for $0 \leqslant x < 5$ and evaluate

$$\int_0^5 [x] \, \mathrm{d}x \, .$$

(ii) Sketch the graph of $y = [\mathrm{e}^x]$ for $0 \leqslant x < \ln n$, where n is an integer, and show that

$$\int_0^{\ln n} [\mathrm{e}^x] \, \mathrm{d}x = n \ln n - \ln(n!) \, .$$

Hence show that $n! \geqslant n^n \mathrm{e}^{1-n}$.

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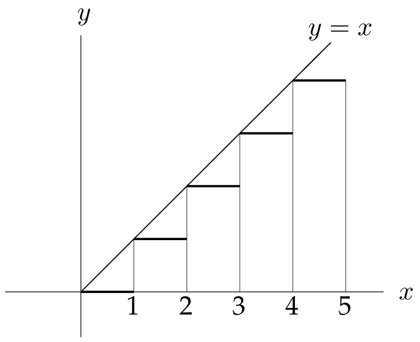
Comments

I had to add a bit to the original question because it was all dressed up with nowhere to go. The question is clearly about estimating $n!$ so I added in the last line, which makes the question a bit longer but not much more difficult. I could have added another part, but I thought that you would probably add it yourself: you will easily spot, once you have drawn the graphs, that a similar result for $n!$ with the inequality reversed can be obtained by considering rectangles the tops of which are above the graph of e^x instead of below it. You therefore end up with a nice sandwich inequality for $n!$.

Stirling (1692–1770) proved in 1730 that $n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} \mathrm{e}^{-n}$ for large n. This was a brilliant result — even reading his book, it is hard to see where he got $\sqrt{2\pi}$ from (especially as he wrote in Latin). Then he went on to obtain the approximation in terms of an infinite series; the expression above is just the first term.

Solution to problem 38

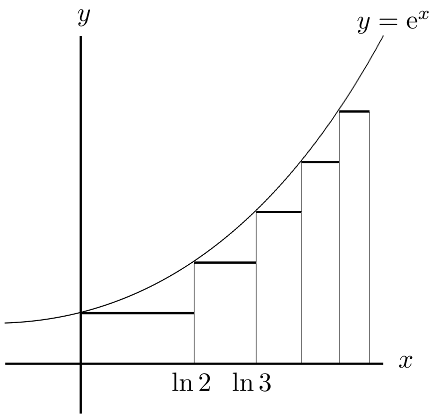
(i)



The graph of $[x]$ consists of the horizontal parts of an ascending staircase with 5 stairs, the lowest at height 0, each of width 1 unit and rising 1 unit.
The integral is the sum of the areas of the rectangles shown in the figure:

$$\text{area} = 0 + 1 + 2 + 3 + 4 = 10\,.$$

(ii)



The graph of $[e^x]$ is also a staircase: the height of each stair is 1 unit and the width decreases as x increases because the gradient of e^x increases. It starts at height 1 and ends at height $(n-1)$.
The value of $[e^x]$ changes from $k-1$ to k when $e^x = k$ i.e. when $x = \ln k$. Thus $[e^x] = k$ when $\ln k \leqslant x < \ln(k+1)$ and the area of the corresponding rectangle is $k\big(\ln(k+1) - \ln k\big)$.
The total area under the curve is therefore

$$1(\ln 2 - \ln 1) + 2(\ln 3 - \ln 2) + \cdots + (n-1)\big(\ln n - \ln(n-1)\big)$$

which you can rearrange to obtain $n \ln n - \ln(n!)$ as required.

For the last part, note that $[e^x] \leqslant e^x$ (this is clear from the definition) so

$$\int_0^{\ln n} [e^x]\,dx \leqslant \int_0^{\ln n} e^x\,dx = n - 1 \quad \text{i.e.} \quad n \ln n - \ln(n!) \leqslant n - 1\,.$$

Taking exponentials gives the required result.

Post-mortem

Usually when you draw a graph of a discontinuous function, you should specify at the jump whether the function takes the upper or lower value. For example $[x]$ takes the value 1 at $x = 1$, which is the upper value. This can be achieved by putting (say) a circle round the upper or lower point, as appropriate. I didn't bother on the above graphs, because it doesn't affect the value of the integral and I didn't want to clutter up the graphs.

Considering $[e^x + 1]$ instead of $[e^x]$ gives rectangles above the graph of $y = e^x$ rather than below. The calculations are roughly the same, so you should easily arrive at $n! \leqslant n^{n+1}e^{1-n}$. We have therefore proved that

$$n^n e^{1-n} \leqslant n! \leqslant n^{n+1} e^{1-n}\,.$$

This gives a pretty good (given the rather elementary method at our disposal) approximation for $n!$ for large n.