

Problem 37: Irrationality of e

(✓✓)

For each positive integer n , let

$$a_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots,$$

$$b_n = \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots.$$

- (i) Show that $b_n = 1/n$.
- (ii) Deduce that $0 < a_n < 1/n$.
- (iii) Show that $a_n = n!e - [n!e]$, where $[x]$ denotes the integer part of x .
- (iv) Hence show that e is irrational.

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Comments

Each part of this looks horrendously difficult, but it doesn't take much thought to see what is going on. If you haven't come across the concept of *the integer part* some examples should make it clear: $[21.25] = 21$, $[\pi] = 3$, $[1.999999] = 1$ and $[2] = 2$.

For part (iii), you need to know the series for the number e , which is $\sum_{n=0}^{\infty} 1/n!$.

If you are stumped by the last part, just remember the definition of the word *irrational*: x is rational if and only if it can be expressed in the form p/q where p and q are integers; if x is not rational, it is irrational. Then look for a proof by contradiction ('Suppose that x is rational ...').

The first proof that e is irrational was given by Euler in 1736. He also named the number e , though he didn't 'invent' it; e is the basis for natural logarithms and as such was used implicitly by John Napier in 1614. Euler's proof involved the use of continuous fractions. He wrote e as:

$$2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{\dots}}}}$$

the numbers in the denominators forming the infinite pattern 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ... It is known that any such continuous fraction represents an irrational number.

In general, it is not easy to show that numbers are irrational. Johann Lambert showed that π is irrational in 1760, but the simple continued fraction for π is not known. It is still not known if $\pi + e$ is irrational.

The proof that e is irrational given in this question is based on a proof by Fourier²⁶ from about 1815.

²⁶ Joseph Fourier (1768–1830) was a French mathematician and administrator. He is best known for his work on the theory of heat flow. He is generally credited with the discovery of the greenhouse effect. He spent some time in Egypt with Napoleon, and contributed much to the modern study of Egyptology.

Solution to problem 37

(i) Although the series for b_n does not at first sight look tractable, it is in fact just a geometric progression: the first term is $1/(n+1)$ and the common ratio is also $1/(n+1)$. Thus

$$b_n = \frac{1}{n+1} \left(\frac{1}{1 - 1/(n+1)} \right) = \frac{1}{n}.$$

(ii) Each term (after the first) of a_n is less than the corresponding term in b_n , so $a_n < b_n = 1/n$.

(iii) Multiplying the series for e by $n!$ gives

$$n!e = n! + n! + \frac{1}{2}n! + \cdots + 1 + a_n$$

and the result follows because $a_n < 1$ and all the other terms on the right hand side of the above equation are integers.

(iv) We use proof by contradiction. Suppose that there exist integers k and m such that $e = k/m$. Then $m!e$ is certainly an integer. But if $m!e$ is an integer then $[m!e] = m!e$, which contradicts the result of part (iii) since we know that $a_m \neq 0$ (it is obvious from the definition that $a_m > 0$).

Post-mortem

Well, that was short!

There were, however, two key steps which were not difficult in themselves but were not easy to find in the context of this question.

The first was to recognise that b_n is, for each n , a simple geometric progression. If it had been presented in the form $b = r + r^2 + r^3 + \cdots$ it would have been immediately recognisable. Somehow, the fact that the ratio r is given in terms of n — or, even worse, $n+1$ — makes it difficult to spot; not least because n is fixed for each series instead of labelling the terms of an individual series.

The second key step, in part (iv), was to assume that $e = k/m$ and show that $m!e$, **not** me , cannot be an integer. It seems so wasteful when it would suffice to show that me is not an integer. We seem to be making a great deal of extra work for ourselves and that profligacy, unusual in mathematics, might well have thrown you off the scent despite the clear signal from part (iii).

I don't regard the idea of using proof by contradiction as a difficult step in part (iv). An irrational number is by definition a number that is not rational, so the idea of a proof by contradiction should jump out at you. Indeed, the standard example, the proof of the irrationality of $\sqrt{2}$, is by contradiction.²⁷

²⁷ Suppose $p\sqrt{2} = q$. Then $2p^2 = q^2$. Writing p and q as the product of primes gives the contradiction since there are an odd number of powers of 2 one side of this equation and an even number on the other.