Problem 35: Roots of a cubic equation (✓✓✓)

Consider the cubic equation

$$x^3 - px^2 + qx - r = 0\ ,$$

where $p \neq 0$ and $r \neq 0$.

(i) If the three roots can be written in the form ak^{-1}, a and ak for some constants a and k, show that one root is q/p and that $q^3 - rp^3 = 0$.

(ii) If $r = q^3/p^3$, show that q/p is a root and that the product of the other two roots is $(q/p)^2$. Deduce that the roots are in geometric progression.

(iii) Find a necessary and sufficient condition involving p, q and r for the roots to be in arithmetic progression.

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Comments

The Fundamental Theorem of Algebra says that a polynomial of degree n can be written as the product of n linear factors, so we can write

$$x^3 - px^2 + qx - r = (x-\alpha)(x-\beta)(x-\gamma)\,, \qquad (*)$$

where α, β and γ are the roots of the equation $x^3 - px^2 + qx - r = 0$. The basis of this question is the comparison between the left hand side of $(*)$ and the right hand side, multiplied out, of $(*)$. Some of the roots may not be real, but you don't have to worry about that here.

The 'necessary and sufficient' in part (iii) looks a bit forbidding, but if you just repeat the steps of parts (i) and (ii), it is straightforward.

Solution to problem 35

(i) We have

$$(x - ak^{-1})(x - a)(x - ak) \equiv x^3 - px^2 + qx - r$$

i.e.

$$x^3 - a(k^{-1} + 1 + k)x^2 + a^2(k^{-1} + 1 + k)x - a^3 = x^3 - px^2 + qx - r\ .$$

Thus $p = a(k^{-1} + 1 + k)$, $q = a^2(k^{-1} + 1 + k)$, and $r = a^3$. Dividing gives $q/p = a$, which is a root, and $q^3/p^3 = a^3 = r$ as required.

(ii) Set $r = q^3/p^3$. Substituting $x = q/p$ into the cubic shows that it is a root:

$$(q/p)^3 - p(q/p)^2 + q(q/p) - (q/p)^3 = 0\ .$$

Since q/p is one root, and the product of the three roots is q^3/p^3 ($= r$ in the original equation), the product of the other two roots must be q^2/p^2. The two roots can therefore be written in the form $k^{-1}(q/p)$ and $k(q/p)$ for some number k , which shows that the three roots are in geometric progression.

(iii) The three roots are in arithmetic progression if and only if they are of the form $(a-d)$, a and $(a+d)$. (I have followed the lead of part (i) in using this form rather than a, $a + d$, $a + 2d$.)

If the roots are in this form, then

$$\big(x - (a - d)\big)\big(x - a\big)\big(x - (a + d)\big) \equiv x^3 - px^2 + qx - r$$

i.e.

$$x^3 - 3ax^2 + (3a^2 - d^2)x - a(a^2 - d^2) \equiv x^3 - px^2 + qx - r\ .$$

Thus $p = 3a$, $q = 3a^2 - d^2$, $r = a(a^2 - d^2)$. A necessary condition is therefore $r = (p/3)(q - 2p^2/9)$. Note that one root is $p/3$.

Conversely, if $r = (p/3)(q - 2p^2/9)$, the equation is

$$x^3 - px^2 + qx - (p/3)(q - 2p^2/9) = 0\ .$$

We can verify that $p/3$ is a root by substitution. Since the roots sum to p and one of them is $p/3$, the others must be of the form $p/3 - d$ and $p/3 + d$ for some d. They are therefore in arithmetic progression.

A necessary and sufficient condition for the roots to be in arithmetic progression is therefore $r = (p/3)(q - 2p^2/9)$.

Post-mortem

As usual, it is a good idea to give a bit of thought to the conditions given, namely $p \neq 0$ and $r \neq 0$.

Clearly, if the roots are in geometric progression, we cannot have a zero root. We therefore need $r \neq 0$. However, we don't need the condition $p \neq 0$. If the roots are in geometric progression with $p = 0$, then $q = 0$, but there is no contradiction: the roots are $be^{-2\pi i/3}$, b and $be^{2\pi i/3}$ where $b^3 = r$, and these are certainly in geometric progression. So the necessary and sufficient conditions are $r \neq 0$ and $p^3 r = q^3$.

Neither of these conditions is required if the roots are in arithmetic progression.

Therefore, the condition $p \neq 0$ is only there as a convenience — one thing less for you to worry about. The condition $r \neq 0$ should really have been given only for the first part. It should be said that this sort of question is incredibly difficult to word, which is why the examiner was a bit heavy handed with the conditions.