

Problem 34: Trig. sum

(✓✓✓)

Prove that

$$\sum_{k=0}^n \sin k\theta = \frac{\cos \frac{1}{2}\theta - \cos(n + \frac{1}{2})\theta}{2 \sin \frac{1}{2}\theta} . \quad (*)$$

(i) Deduce that, for $n \geq 1$,

$$\sum_{k=0}^n \sin \left(\frac{k\pi}{n} \right) = \cot \left(\frac{\pi}{2n} \right) .$$

(ii) By differentiating (*) with respect to θ , or otherwise, show that, for $n \geq 1$,

$$\sum_{k=0}^n k \sin^2 \left(\frac{k\pi}{2n} \right) = \frac{(n+1)^2}{4} + \frac{1}{4} \cot^2 \left(\frac{\pi}{2n} \right) .$$

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Comments

The very first part can be done by multiplying both sides of (*) by $\sin \frac{1}{2}\theta$ and using the identity

$$2 \sin A \sin B = \cos(B - A) - \cos(B + A) .$$

It can also be done by induction (worth a try even if you do it by the above method) or by considering the imaginary part of $\sum \exp(ik\theta)$ (summing this as a geometric progression), if you know about de Moivre's theorem.

Before setting pen to paper for part (ii), it pays to think very hard about simplifications of the right-hand side of (*) that might make the differentiation more tractable – perhaps bearing in mind the calculations required for part (i).

In the original question, you were asked to show that, for large n ,

$$\sum_{k=0}^n \sin \left(\frac{k\pi}{n} \right) \approx \frac{2n}{\pi}$$

and

$$\sum_{k=0}^n k \sin^2 \left(\frac{k\pi}{2n} \right) \approx \left(\frac{1}{4} + \frac{1}{\pi^2} \right) n^2 ,$$

using the approximations, valid for small θ , $\sin \theta \approx \theta$ and $\cos \theta \approx 1 - \frac{1}{2}\theta^2$. Of course, these results follow quickly from the exact results; but if you only want approximate results you can save yourself a bit of work by approximating early to avoid doing some of the trigonometric calculations. I thought that the exact results were nicer than the approximate results so I changed the question a bit for this book (though I felt a bit guilty, because approximations are an important part of mathematics).

Solution to problem 34

For the first part, we will show that $\sum_{k=0}^n 2 \sin k\theta \sin \frac{1}{2}\theta = \cos \frac{1}{2}\theta - \cos(n + \frac{1}{2})\theta$. Starting with the left hand side, we have:

$$\sum_{k=0}^n 2 \sin k\theta \sin \frac{1}{2}\theta = \sum_{k=0}^n [-\cos(k + \frac{1}{2})\theta + \cos(k - \frac{1}{2})\theta]$$

which gives the result immediately, since almost all the terms in the sum cancel. (Write out a few terms if you are not certain of this.)

(i) Let $\theta = \pi/n$. Then

$$\sum_{k=0}^n \sin\left(\frac{k\pi}{n}\right) = \frac{\cos \frac{1}{2}(\pi/n) - \cos(\pi + \frac{1}{2}(\pi/n))}{2 \sin \frac{1}{2}(\frac{\pi}{n})} = \frac{\cos \frac{1}{2}(\pi/n) + \cos \frac{1}{2}(\pi/n)}{2 \sin \frac{1}{2}(\pi/n)} = \cot \frac{1}{2}(\pi/n)$$

as required.

(ii) Differentiating the left hand side of (*) and using a double angle trig. formula gives

$$\sum_{k=0}^n k \cos k\theta = \sum_{k=0}^n k (1 - 2 \sin^2 \frac{1}{2}k\theta) = \frac{1}{2}n(n+1) - 2 \sum_{k=0}^n k \sin^2 \frac{1}{2}k\theta. \quad (\dagger)$$

Before attempting to differentiate the right hand side of (*) it is a good idea to write it in a form that gets rid of some of the fractions. Omitting for the moment the factor $1/2$, we have

$$\text{RHS of } (*) = (1 - \cos n\theta) \cot \frac{1}{2}\theta + \sin n\theta$$

and differentiating gives $-\frac{1}{2}(1 - \cos n\theta) \operatorname{cosec}^2 \frac{1}{2}\theta + n \sin n\theta \cot \frac{1}{2}\theta + n \cos n\theta$. Now setting $\theta = \pi/n$ gives

$$-\operatorname{cosec}^2 \frac{1}{2}(\pi/n) - n. \quad (\ddagger)$$

Setting $\theta = \pi/n$ in equation (†) and comparing with (‡) (remembering that there is a factor of $1/2$ missing) gives the required result.

Post-mortem

Now that I have had another go at this question it does not seem terribly interesting. At first, I thought I might ditch it. Then I thought that it perhaps was worthwhile: keeping cool under the pressure of the differentiation for part (ii) – it should just be a few careful lines – and getting it out is a good confidence booster. Anyway, now that I have slogged²⁴ through it, I am going to leave it in.

[Next day] I recall now that the reason that I included this question in the first place was because in its original form (with approximate answers as given in the comments section above), the result for part (ii) can be obtained very quickly from part (i) by differentiating both sides with respect to π and using a double angle formula. Try it!

Unfortunately, you'd have a hard time justifying this approach, and anyway it is not going to work for the exact result.²⁵

²⁴ I should come clean at this point: I just differentiated (*) on autopilot, without considering whether I could simplify the task by rewriting the formula as suggested in my hint on the previous page. It was hard work. Serves me right for not following my own general advice.

²⁵ For large n we are just adding up a lot of sin curves and it doesn't much matter what the exact value of π is (so we can take it to be a variable); for the exact result, the exact value of π does matter.