

Problem 33: More curve sketching

(✓✓)

The curve C has equation

$$y = \frac{x}{\sqrt{x^2 - 2x + a}},$$

where the square root is positive. Show that, if $a > 1$, then C has exactly one stationary point. Sketch C when (i) $a = 2$ and (ii) $a = 1$.

1999 Paper II

Comments

You have to be sure of the definition of $\sqrt{x^2 - 2x + a}$ to do this question. You might think that it is ambiguous, because the square root could be positive or negative, but by convention it means the *positive* square root. Thus $\sqrt{x^2} = |x|$ (not x).

For the sketches, you just need the position of any stationary points, any other interesting points that the graph passes through, behaviour as $x \rightarrow \pm\infty$ and any vertical asymptotes. It should not be necessary in such a simple case to establish the nature of the stationary point(s). The sketches are not particularly difficult, but they do need thought because they turn out to be rather different from the sketches of polynomials or trigonometric functions that you are probably used to.

You should at some stage think about the conditions given on a . In fact, I doubt if many of you will want to leave it there. It is clear that what the examiner would really have liked you to do is to sketch C in the different cases that arise according to the value of a but was told that this would be too long and/or difficult for the exam; but I am sure that this is what you will try to do (or maybe what you have already done by the time you read this).

Please do not use your calculator (or Wolfram Alpha or Geogebra or Desmos²³, etc) for these sketches, except perhaps to check your answers: there is no point.

²³ <https://www.desmos.com/calculator> is brilliant: you can use a slider to change the value of a ; but please don't.

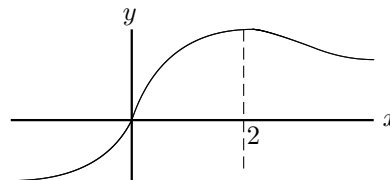
Solution to problem 33

First differentiate to find the stationary points:

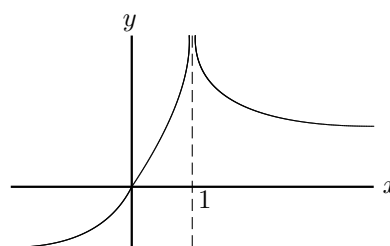
$$\frac{dy}{dx} = \frac{(x^2 - 2x + a) - x(x - 1)}{(x^2 - 2x + a)^{\frac{3}{2}}} = \frac{a - x}{(x^2 - 2x + a)^{\frac{3}{2}}}$$

which is only zero when $x = a$. I've used the quotient rule, but the product rule is just as good.

(i) When $a = 2$, the stationary point is at $(2, \sqrt{2})$ (which can be seen to be a maximum by evaluating the second derivative at $x = a$, though this is not necessary). The curve passes through $(0, 0)$. As $x \rightarrow \infty, y \rightarrow 1$; as $x \rightarrow -\infty, y \rightarrow -1$.



(ii) When $a = 1$, we can rewrite y as $\frac{x}{|x - 1|}$. The modulus signs arise because the square root is taken to be positive (or zero — but that can be discounted here since it is in the denominator). The graph is as before, except that the maximum point has been stretched into a vertical asymptote (like a volcanic plug) at $x = 1$. Note that if $|x|$ is much larger than a , it doesn't really matter whether $a = 1$ or $a = 2$.



Post-mortem

The first thing to do after finishing the question is to try to understand the role of the parameter a . Part (ii) above is the key. Writing

$$f(x) = x^2 - 2x + a = (x - 1)^2 + a - 1,$$

we see that $f(x) > 0$ for all x if $a > 1$ but if $a < 1$ there are values of x for which $f(x) < 0$. The borderline value is $a = 1$, for which $f(x) \geq 0$ and $f(1) = 0$.

If $a < 1$, we find that $f(x) = 0$ when $x = 1 \pm \sqrt{1 - a}$ and $f(x) < 0$ between these two values of x . The significance of this is that the square root in the denominator of the function we are trying to sketch is imaginary, so there is a gap in the graph between $x = 1 - \sqrt{1 - a}$ and $x = 1 + \sqrt{1 - a}$. There are vertical asymptotes at these values.

Is that the whole story? Well, no. We now have to decide whether the two significant points of our graph, namely $x = 0$ where the curve crosses the axis and $x = a$ where the curve has a maximum, lie inside or outside the forbidden zone $1 - \sqrt{1 - a} \leq x \leq 1 + \sqrt{1 - a}$. This turns out to depend on whether $a < 0$ or $a = 0$ or $0 < a < 1$. The graph will look qualitatively different in each case, so there is still quite a lot of work to do in this $a < 1$ case! We can see why it was excluded from the original question.