

Problem 32: Integration by substitution

(✓✓)

Find $\frac{dy}{dx}$ if

$$y = \frac{ax + b}{cx + d} . \quad (*)$$

By using changes of variable of the form (*), or otherwise, show that

$$\int_0^1 \frac{1}{(x+3)^2} \ln \left(\frac{x+1}{x+3} \right) dx = \frac{1}{6} \ln 3 - \frac{1}{4} \ln 2 - \frac{1}{12} ,$$

and evaluate the integrals

$$\int_0^1 \frac{1}{(x+3)^2} \ln \left(\frac{x^2+3x+2}{(x+3)^2} \right) dx \quad \text{and} \quad \int_0^1 \frac{1}{(x+3)^2} \ln \left(\frac{x+1}{x+2} \right) dx .$$

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Comments

You will find that the change of variable in each case is clearly signalled: it is really only the denominator of (*) that matters.

For the first integral, you do the obvious thing, but for the second and third integrals you have to be quite ingenious to get the argument of the logs in a suitable form. Once you have got the idea for the second integral, you should be able to see the connection with the third integral, but it would be hard to do the third integral without having done the second. That's what I like about this question: one thing leads to another.

I have written below the version of this question that was proposed by the setter, because I thought you would be interested to see how a question evolves.

By changing to the variable y defined by

$$y = \frac{2x-3}{x+1} ,$$

evaluate the integral

$$\int_2^4 \frac{2x-3}{(x+1)^3} \ln \left(\frac{2x-3}{x+1} \right) dx .$$

Evaluate the integral

$$\int_9^{25} (2z^{-\frac{3}{2}} - 5z^{-2}) \ln (2 - 5z^{-\frac{1}{2}}) dz .$$

Note in particular the way that the ideas in the final draft are closely knit and better structured: the first change of variable is strongly signalled but not explicit and the following parts, though based on the same idea, require increasing ingenuity. The first draft required quite a jump to evaluate the second integral.

Note also that the final integral of the first draft has an unpleasant contrived appearance, whereas the integrals of the final draft are rather pleasing: beauty matters to mathematicians.

Solution to problem 32

Differentiating gives $\frac{dy}{dx} = \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}$.

For the first integral, set $y = \frac{x+1}{x+3}$:

$$\int_0^1 \frac{1}{(x+3)^2} \ln\left(\frac{x+1}{x+3}\right) dx = \frac{1}{2} \int_{1/3}^{1/2} \ln y dy = \frac{1}{2} \left[y \ln y - y \right]_{1/3}^{1/2},$$

which gives the required answer. (The integral of $\ln y$ is a standard integral; it can be done by parts, first substituting $z = \ln y$, if you like.)

The second integral can be expressed as the sum of two integrals of the same form as the first integral, since

$$\ln\left(\frac{x^2+3x+2}{(x+3)^2}\right) = \ln\left(\frac{(x+1)(x+2)}{(x+3)^2}\right) = \ln\left(\frac{x+1}{x+3}\right) + \ln\left(\frac{x+2}{x+3}\right).$$

We have already done the first of these integrals. Using the substitution $y = \frac{x+2}{x+3}$ in the second of these integrals gives

$$\int_{2/3}^{3/4} \ln y dy = \frac{17}{12} \ln 3 - \frac{13}{6} \ln 2 - \frac{1}{12}.$$

For the final integral, note that

$$\ln\left(\frac{x+1}{x+2}\right) = \ln\left(\frac{x+1}{x+3}\right) - \ln\left(\frac{x+2}{x+3}\right),$$

so the required integral is the difference of the two integrals that we summed in the previous part, i.e.

$$-\frac{5}{4} \ln 3 + \frac{23}{12} \ln 2.$$

Post-mortem

I hope you had time to try the original version of this question, given in the comments above. The last part is a nice puzzle.

For the first integral of the original version, the change of variable gives

$$\frac{1}{5} \int_{\frac{1}{3}}^1 y \ln y dy = \frac{1}{90} \ln 3 - \frac{2}{45}.$$

I checked this using Wolfram Alpha. For the second integral, you have to guess the substitution. There are plenty of clues, but the most obvious place to start is the log. The argument of the log can be written as

$$\frac{2\sqrt{z}-5}{\sqrt{z}}.$$

It is now a bit of a leap in the dark (too much of a leap, I thought), but if we decide to convert this exactly into the previous integral, we should take $z = (x+1)^2$. And, magic (or contrived STEP question), it works, giving the first integral almost exactly (note especially that the limits transform as they should). The only difference is a factor of 2 so the answer is twice the previous answer.

A change of variable of the form (*) is called a *linear fractional* or *Möbius* transformation. It is of great importance in the theory of the geometry of the complex plane.