

Problem 31: The harmonic series

(✓✓✓)

The function f satisfies $0 \leq f(t) \leq K$ when $0 \leq t \leq 1$. Explain by means of a sketch, or otherwise, why

$$0 \leq \int_0^1 f(t) dt \leq K.$$

By considering $\int_0^1 \frac{t}{n(n-t)} dt$ or otherwise show that, if $n > 1$, then

$$0 \leq \ln \left(\frac{n}{n-1} \right) - \frac{1}{n} \leq \frac{1}{n-1} - \frac{1}{n}$$

and deduce that

$$0 \leq \ln N - \sum_{n=2}^N \frac{1}{n} \leq 1.$$

Deduce that $\sum_{n=1}^N \frac{1}{n} \rightarrow \infty$ as $N \rightarrow \infty$.

Noting that $2^{10} = 1024$, show also that if $N < 10^{30}$ then $\sum_{n=1}^N \frac{1}{n} < 101$.

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Comments

Quite a lot of different ideas are required for this question; hence ✓✓✓.

The first hurdle is to decide what the constant K in the first part has to be in order to make the second part work. You might try to maximise the integrand using calculus, but that would be the wrong thing to do (you should find if you do this that the integrand has no turning point in the given range: it increases throughout the range).

The next hurdle is the third displayed equation, which follows from the preceding result. Then there is still more work to do.

It is not at all obvious that the series $\sum_{n=1}^N \frac{1}{n}$ (which is called the *harmonic series*) tends to infinity as N increases. There are easier ways of proving this than the method used in this question, but this way tells us two interesting things. First it tells us that the sum increases very slowly indeed: the first 10^{30} terms only get to 100. Second it tells us that

$$0 \leq \sum_{n=1}^N \frac{1}{n} - \ln N \leq 1$$

for all N . (This is the third displayed equation in the question slightly rewritten.) Not only does the sum diverge, it does so logarithmically. In fact, in the limit $N \rightarrow \infty$,

$$\sum_{n=1}^N \frac{1}{n} - \ln N \rightarrow \gamma$$

where γ is *Euler's constant*. Its value is about $\frac{1}{2}$.

Solution to problem 31

Any sketch showing a squiggly curve all of which lies beneath the line $x = K$ and above the x -axis will do: area under curve (the integral) is less than the area of the rectangle (K).

Let's start the next part in the obvious way, by evaluating the integral.

Noting that $\frac{t}{n-t} = \frac{n}{n-t} - 1$, we have:

$$\int_0^1 \frac{t}{n(n-t)} dt = \int_0^1 \left(\frac{1}{n-t} - \frac{1}{n} \right) dt = \ln \left(\frac{n}{n-1} \right) - \frac{1}{n}.$$

The largest value of the integrand in the interval $[0, 1]$ occurs at $t = 1$, since the numerator increases as t increases and the denominator decreases as t increases (remember that $n > 1$). Using the result of the first paragraph gives

$$0 \leq \ln \left(\frac{n}{n-1} \right) - \frac{1}{n} \leq \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

Summing both sides from 2 to N and cancelling lots of terms in pairs gives

$$0 \leq \ln N - \sum_2^N \frac{1}{n} \leq 1 - \frac{1}{N}. \quad (*)$$

Note that $1 - \frac{1}{N} < 1$. Since $\ln N \rightarrow \infty$ as $N \rightarrow \infty$, so also must $\sum_2^N \frac{1}{n}$ (it differs from the logarithm by less than 1).

Finally, rearranging the first inequality in $(*)$ gives $\sum_2^N \frac{1}{n} \leq \ln N$, i.e. $\sum_1^N \frac{1}{n} \leq \ln N + 1$. Setting $N = 10^{30}$, and using the inequalities $1000 < 1024$ and $e > 2$ (or $\ln 2 < 1$) gives:

$$\sum_1^{10^{30}} \frac{1}{n} \leq \ln(10^{30}) + 1 < \ln(1024^{10}) + 1 = 100 \ln 2 + 1 < 100 \ln e + 1 = 100 + 1.$$

Post-mortem

This question seems very daunting at first, because you are asked to prove a sequence of completely unfamiliar results. However, you should learn from this question that if you keep cool and follow the hints, explicit and implicit, you can achieve some surprisingly sophisticated results. You might think that this situation is very artificial: in real life, you do not receive hints to guide you. But often in mathematical research, hints *are* buried deep in the problem if only you can recognise them.