

Problem 30: Integer solutions of cubics

(✓✓✓)

- (i) Show that, if
- m
- is an integer such that

$$(m-3)^3 + m^3 = (m+3)^3, \quad (*)$$

then m^2 is a factor of 54. Deduce that there is no integer m which satisfies the equation (*).

- (ii) Show that, if
- n
- is an integer such that

$$(n-6)^3 + n^3 = (n+6)^3, \quad (**)$$

then n is even. By writing $n = 2m$ deduce that there is no integer n which satisfies the equation (**).

- (iii) Show that, if
- n
- is an integer such that

$$(n-a)^3 + n^3 = (n+a)^3, \quad (***)$$

where a is a non-zero integer, then n is even and a is even. Deduce that there is no integer n which satisfies the equation (** *).

1998 Paper II

Comments

I slightly simplified the first two parts of the question, which comprised the whole of the original STEP question, and added the third part. This last part is conceptually tricky and very interesting: hence the ✓✓✓ rating.

It is a fairly standard technique in this sort of number theoretic problem to investigate whether the equation *balances*: for example, if the left-hand side is even, then the right-hand side must also be even. If this fails, then the equation can have no solutions.

Solution to problem 30

(i) Simplifying gives $m^2(m - 18) = 54$.

Both m^2 and $(m - 18)$ must divide 54, which is impossible since the only squares that divide 54 are 1 and 9, and neither $m = 1$ nor $m = 3$ satisfies $m^2(m - 18) = 54$.

You could also argue that $m - 18$ must be positive so $m \geq 19$ and $m^2 \geq 361$ which is a contradiction.

(ii) The easiest method is a proof by contradiction. Suppose therefore that n is odd. The two terms on the left-hand side are both odd, which means that the left-hand side is even. But the right-hand side is odd so the equation cannot balance if n is odd.

Setting $n = 2m$ gives

$$(2m - 6)^3 + (2m)^3 = (2m + 6)^3.$$

Taking a factor of 2^3 out of each term leaves $(m - 3)^3 + m^3 = (m + 3)^3$ which is the same as the equation that was shown in part (i) to have no solutions.

(iii) First we show that n is even, dealing with the cases a odd and a even separately. The first case is n odd and a odd. In that case, $(n - a)^3$ is even, n^3 is odd and $(n + a)^3$ is even, so the equation does not balance. In the second case, n is odd and a is even, the three terms are all odd and again the equation does not balance. Therefore n cannot be odd.

Next, we investigate the case n is even and a odd. This time the three terms are odd, even and odd respectively so there is no contradiction. But multiplying out the brackets and simplifying gives

$$n^2(n - 6a) = 2a^3.$$

Since n is even the left-hand side is divisible by 4, because of the factor n^2 . That means that a^3 is divisible by 2 which cannot be the case since a is odd.

Now that we know n and a are both even, we can follow the method used in part (ii) and set $n = 2m$ and $a = 2b$. This gives

$$(2m - 2b)^3 + (2m)^3 = (2m + 2b)^3$$

from which a factor of 2^3 can be cancelled from each term. Thus m and b satisfy the same equation as n and a . They are therefore both even and we can repeat the process.

Repeating the process again and again will eventually result in an integer that is odd which will therefore not satisfy the equation that it is supposed to satisfy: a contradiction. There is therefore no integer n that satisfies $(*)$.

Post-mortem

The last proof is an example of what is known as the *method of infinite descent*. It was used by Fermat (1601 – 1665) to prove special cases of his Last Theorem²², which of course is exactly what you are doing in this question. The method was probably invented by him and his faith in it sometimes led him astray. It is even possible that he thought he could use it to prove his Last Theorem in full. In fact, the proof of this theorem was only given in 1994 by Andrew Wiles; and it is 150 pages of pretty incomprehensible modern mathematics.

²² The theorem says that the equation $x^n + y^n = z^n$, where x, y, z and n are positive integers, can only hold if $n = 1$ or 2 .