

Problem 29: Telescoping series

(✓✓)

- (i) Show that the sum
- S_N
- of the first
- N
- terms of the series

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \cdots + \frac{2n-1}{n(n+1)(n+2)} + \cdots$$

is

$$\frac{1}{2} \left(\frac{3}{2} + \frac{1}{N+1} - \frac{5}{N+2} \right).$$

What is the limit of S_N as $N \rightarrow \infty$?

- (ii) The numbers
- a_n
- are such that

$$\frac{a_n}{a_{n-1}} = \frac{(n-1)(2n-1)}{(n+2)(2n-3)}.$$

Find an expression for $\frac{a_n}{a_1}$ and hence, or otherwise, evaluate $\sum_{n=1}^{\infty} a_n$ when $a_1 = \frac{2}{9}$.

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Comments

If you haven't the faintest idea how to do the sum, then look at the first line of the solution; but don't look without first having had a long think about it, picking up ideas from the form of the given answer. Limits form an important part of first year university mathematics. The definition of a limit is one of the basic ideas in *analysis*, which is the rigorous study of calculus. At the end of part (i), no such definition is needed: you just see what happens when N gets very large (some terms get very small and eventually go away).

Part (ii) looks as if it might be some new idea. Since this is STEP, you will probably realise that the new series must be closely related to the series in part (i). The peculiar choice for $a_1 (= \frac{2}{9})$ should make you suspect that the sum will come out to some nice round number (not in fact round in this case, but straight and thin).

Having decided how to do the first part, please don't use the 'cover up' rule unless you understand why it works: mathematics at this level is not a matter of applying learned recipes. See the post-mortem for more thoughts on this matter.

Solution to problem 29

The given answer to the sum suggests partial fractions. It is difficult to think of any other way of starting, so let's convert the general term of the series to partial fractions in the hope that something good might happen. Set

$$\frac{2n-1}{n(n+1)(n+2)} \equiv \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}, \quad (*)$$

then use your favourite method to find $A = -\frac{1}{2}$, $B = 3$ and $C = -\frac{5}{2}$. Note that $A + B + C = 0$. The series can now be written

$$\begin{aligned} & \left(\frac{A}{1} + \frac{B}{2} + \frac{C}{3} \right) + \left(\frac{A}{2} + \frac{B}{3} + \frac{C}{4} \right) + \left(\frac{A}{3} + \frac{B}{4} + \frac{C}{5} \right) + \\ & + \cdots + \left(\frac{A}{N-2} + \frac{B}{N-1} + \frac{C}{N} \right) + \left(\frac{A}{N-1} + \frac{B}{N} + \frac{C}{N+1} \right) + \left(\frac{A}{N} + \frac{B}{N+1} + \frac{C}{N+2} \right). \end{aligned} \quad (**)$$

Now we collect up terms with the same denominators and find that all the terms in the series cancel, except those with denominators 1, 2, $N+1$ and $N+2$. These exceptions sum to the required answer.

The limit as $N \rightarrow \infty$ is $\frac{3}{4}$ since the other two terms obviously tend to zero.

For part (ii), we note that $\frac{a_n}{a_{n-1}} = \frac{b_n}{b_{n-1}}$, where b_n is the general term of the series in part (i). Thus

$$\frac{a_n}{a_1} = \frac{b_n}{b_1} \quad \text{and} \quad \sum_1^\infty a_n = \frac{a_1}{b_1} \sum_1^\infty b_n = \frac{2/9}{1/6} \times \frac{3}{4} = 1.$$

Alternatively, we can write out the n th term explicitly:

$$\begin{aligned} a_n &= \frac{(n-1)(2n-1)}{(n+2)(2n-3)} a_{n-1} = \frac{(n-1)(2n-1)}{(n+2)(2n-3)} \frac{(n-2)(2n-5)}{(n+1)(2n-7)} a_{n-2} \\ &= \frac{(n-1)(2n-1)}{(n+2)(2n-3)} \frac{(n-2)(2n-5)}{(n+1)(2n-7)} \frac{(n-3)(2n-9)}{(n)(2n-11)} \cdots \frac{5 \times 11}{8 \times 9} \frac{4 \times 9}{7 \times 7} \frac{3 \times 7}{6 \times 5} \frac{2 \times 5}{5 \times 3} \frac{1 \times 3}{4 \times 1} a_1 \\ &= \frac{2n-1}{n(n+1)(n+2)} \frac{3 \times 2 \times 1}{1} a_1 = \frac{12}{9} \frac{2n-1}{n(n+1)(n+2)}, \end{aligned}$$

all other terms cancelling. Now using the result of the first part gives $\sum_1^\infty a_n = 1$.

Post-mortem

A small point of technique: equation (**) was made much clearer (and it saved writing) to stick with A , B and C instead of using $-\frac{1}{2}$, 3 and $-\frac{5}{2}$. The method would not depend on the arithmetic values of these constants.

There are various methods for finding A , B and C in (*).

One is to set $n = 1$, $n = 2$ and $n = 3$ consecutively and obtain three simultaneous equations.

Another is to multiply up and simplify, giving $2n-1 = (A+B+C)n^2 + (3A+2B+C)n + 2A$. You then equate coefficients of powers of n .

A better way is to multiply up without simplifying, giving $2n-1 = A(n+1)(n+2) + Bn(n+2) + Cn(n+1)$. You then choose values for n that give quick results: for example, setting $n = -1$ gives $B = 3$ immediately. This is of course the method behind the iniquitous 'cover-up rule'. Note that the 'equivalence' sign, $\equiv$, indicates an identity (something that holds for all values of n) rather than an equation to solve for n , so n doesn't have to be a positive integer (you could set $n = -\frac{1}{2}$ if you fancied it).