Problem 27: Relation between coefficients of quartic for real roots(✓✓)

In this question you may assume that, if $k_1, \ldots, k_n$ are distinct positive real numbers, then

$$\frac{1}{n}\sum_{r=1}^{n} k_r > \left(\prod_{r=1}^{n} k_r\right)^{\frac{1}{n}},$$

i.e. their arithmetic mean is greater than their geometric mean.
Suppose that a, b, c and d are positive real numbers such that the polynomial

$$\mathrm{f}(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$$

has four distinct positive roots.

(i) By considering the relationship between the coefficients of f and its roots, show that $c > d$.

(ii) By differentiating f, show that $b > c$.

(iii) Show that $a > b$.

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Comments

This result is both surprising and pleasing.

The question looks difficult, but you don't have to go very far before you come across something to substitute into the given arithmetic mean/geometric mean (AM/GM) inequality.

To obtain the relationship between the coefficients and the roots for part (i), you need to write the quartic equation in the form $(x-p)(x-q)(x-r)(x-s) = 0$.

Watch out for the condition in the given form of the arithmetic/geometric inequality that the numbers are distinct: you will have to show that any numbers (or algebraic expressions) you use in the inequality are distinct.

I puzzled over this question for ages, not understanding the idea behind it[19].

It will be clear from the proof that it can be generalised to any equation of the form

$$x^N + \sum_{k=0}^{N-1} \binom{N}{k} (-a_k)^{N-k} x^k = 0,$$

where the numbers a_k are distinct and positive, which has positive distinct roots.

Eventually, I realised that the result of the question has really nothing to do with quartic equations, though quartic equations provide a neat method of proof. The inequalities, which apply to any positive numbers, form a sequence interpolating between the AM and the GM. These inequalities were first derived by Newton and Maclaurin in the 17th century but don't seem to be very well known now.

[19] I mentioned this in a talk I gave to sixth formers, and later saw irritatingly on www.thestudentroom.co.uk a comment to the effect that 'even Dr Siklos can't do this STEP question'. Of course I could *do* it; I wanted to *understand* it, as I hope you do too.

Solution to problem 27

(i) First write

$$f(x) \equiv (x-p)(x-q)(x-r)(x-s)$$

where p, q, r and s are the four roots of the equation (known to be real, positive and distinct). Multiplying out the brackets and comparing with $x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$ shows that $pqrs = d^4$ and $pqr + qrs + rsp + spq = 4c^3$.

The required result, $c > d$, follows immediately by applying the AM/GM inequality to the positive real numbers pqr, qrs, rsp and spq:

$$c^3 = \frac{pqr + qrs + rsp + spq}{4} > [(pqr)(qrs)(rsp)(spq)]^{1/4} = [p^3q^3r^3s^3]^{1/4} = [d^{12}]^{1/4}.$$

Taking the cube root (c and d are positive) preserves the inequality.

The AM/GM inequality at the beginning of the question is stated only for the case when the numbers are distinct (though in fact it holds provided at least two of the numbers are distinct). To use the inequality as above, we must therefore show that no two of pqr, qrs, rsp and spq are equal. This follows immediately from the fact that the roots are distinct and non-zero. For if, for example, $pqr = qrs$ then $qr(p-s) = 0$ which means that $q = 0$, $r = 0$ or $p = s$, all of which are ruled out.

(ii) The polynomial $f'(x)$ is cubic so it has three zeros (roots). These are at the turning points of $f(x)$, which lie between the zeros of $f(x)$ and are therefore distinct and positive.

Now

$$f'(x) = 4x^3 - 12ax^2 + 12b^2x - 4c^3,$$

so at the turning points

$$x^3 - 3ax^2 + 3b^2x - c^3 = 0.$$

Suppose that the roots of this cubic equation are u, v and w, all real, distinct and positive. Then comparing $x^3 - 3ax^2 + 3b^2x - c^3 = 0$ and $(x-u)(x-v)(x-w)$ shows that $c^3 = uvw$ and $3b^2 = vw + wu + uv$.

The required result, $b > c$, follows immediately by applying the AM/GM inequality to the positive distinct numbers vw, wu and uv.

(iii) Apply similar arguments to $f''(x)/12$.

Post-mortem

The general inequalities mentioned in the comments above are interesting. For example, for four numbers a_1, a_2, a_3 and a_4, we define

$$S_1 = \frac{a_1 + a_2 + a_3 + a_4}{4}, \qquad S_2 = \frac{a_2a_3 + a_3a_1 + a_1a_2 + a_4a_1 + a_4a_2 + a_4a_3}{6},$$

$$S_3 = \frac{a_2a_3a_4 + a_1a_3a_4 + a_1a_2a_4 + a_1a_2a_3}{4} \quad \text{and} \quad S_4 = a_1a_2a_3a_4.$$

Except for the denominators and alternating signs, S_i is the coefficient of x^{4-i} in the equation $(x-a_1)(x-a_2)(x-a_3)(x-a_4) = 0$, but that doesn't matter. Maclaurin's inequalities are

$$S_1 \geqslant S_2^{\frac{1}{2}} \geqslant S_3^{\frac{1}{3}} \geqslant S_4^{\frac{1}{4}}.$$

What is really surprising is that we proved the three stronger inequalities from the apparently weaker AM>GM inequality ($S_1 \geqslant S_4^{\frac{1}{4}}$): a rare example of pulling yourself up by your bootstraps.