

## Problem 26: Simultaneous integral equations

(✓✓)

Let

$$I = \int_0^a \frac{\cos x}{\sin x + \cos x} dx \quad \text{and} \quad J = \int_0^a \frac{\sin x}{\sin x + \cos x} dx ,$$

where  $0 \leq a < \frac{3}{4}\pi$ . By considering  $I + J$  and  $I - J$ , show that  $2I = a + \ln(\sin a + \cos a)$ .

Find also:

(i)  $\int_0^{\frac{1}{2}\pi} \frac{\cos x}{p \sin x + q \cos x} dx$ , where  $p$  and  $q$  are positive numbers;

(ii)  $\int_0^{\frac{1}{2}\pi} \frac{\cos x + 4}{3 \sin x + 4 \cos x + 25} dx$ .

2002 Paper I

## Comments

This is a model for a perfect STEP question: you are told how to do the first part, and you have to adapt the idea on your own for the later parts. Note the structure of the question: there is a ‘stem’ (the first paragraph) containing material that will be useful for both the later parts.

In the examination, candidates who were successful in parts (i) and (ii) nearly always started off with the statement ‘Now let  $I = \dots$  and  $J = \dots$ ’. If you are stuck with part (ii), the choice of significant numbers (3, 4 and 25) should provide a clue.

When you arrive at an answer for part (i), you will of course check that it agrees with the given result in the opening paragraph when  $a = \frac{1}{2}\pi$  and  $p = q = 1$ .

You will no doubt have noticed the restriction  $0 \leq a < \frac{3}{4}\pi$  given in the first paragraph (and also the restrictions on  $p$  and  $q$  in part (i)). You should try to work out its purpose because it might provide an insight into the method of tackling the question, though in this case it doesn’t.

## Solution to problem 26

For the first part, we regard the two integrals essentially as a pair of simultaneous equations, adding and subtracting to simplify them. We have

$$\begin{aligned} I + J &= \int_0^a \frac{\cos x + \sin x}{\sin x + \cos x} dx = \int_0^a dx = a \\ I - J &= \int_0^a \frac{\cos x - \sin x}{\sin x + \cos x} dx = \ln(\cos a + \sin a) \end{aligned}$$

(in the second integral, the numerator is the derivative of the denominator). Adding these equations gives the required expression for  $2I$ .

(i) Similarly, let  $I = \int_0^{\frac{1}{2}\pi} \frac{\cos x}{p \sin x + q \cos x} dx$  and  $J = \int_0^{\frac{1}{2}\pi} \frac{\sin x}{p \sin x + q \cos x} dx$ . Then

$$\begin{aligned} qI + pJ &= \int_0^{\frac{1}{2}\pi} \frac{q \cos x + p \sin x}{p \sin x + q \cos x} dx = \frac{1}{2}\pi \\ pI - qJ &= \int_0^{\frac{1}{2}\pi} \frac{p \cos x - q \sin x}{p \sin x + q \cos x} dx = \ln(p \sin \tfrac{1}{2}\pi + q \cos \tfrac{1}{2}\pi) - \ln(p \sin 0 + q \cos 0) = \ln \frac{p}{q}. \end{aligned}$$

Now we solve these two equations simultaneously for  $I$ :

$$(p^2 + q^2)I = \frac{q\pi}{2} + p \ln \frac{p}{q}.$$

(ii) This time, let  $I = \int_0^{\frac{1}{2}\pi} \frac{\cos x + 4}{3 \sin x + 4 \cos x + 25} dx$  and  $J = \int_0^{\frac{1}{2}\pi} \frac{\sin x + 3}{3 \sin x + 4 \cos x + 25} dx$ . Then

$$\begin{aligned} 4I + 3J &= \int_0^{\frac{1}{2}\pi} \frac{4 \cos x + 3 \sin x + 25}{3 \sin x + 4 \cos x + 25} dx = \frac{\pi}{2} \\ 3I - 4J &= \int_0^{\frac{1}{2}\pi} \frac{3 \cos x - 4 \sin x}{3 \sin x + 4 \cos x + 25} dx = \ln \frac{28}{29}. \end{aligned}$$

Solving simultaneously gives

$$25I = 2\pi + 3 \ln \frac{28}{29}.$$

## Post-mortem

I very much like this question. The first part appeared on STEP Paper I in 1995 (the second part of the 1995 question was an integral completely unrelated to the first part — that wouldn't happen now) and I was completely taken by surprise.

In order to use it again for STEP in 2002, I added parts (i) and (ii). It took me some time to think of a suitable extension. I was disappointed to find that the basic idea is more or less a one-off: there are very few denominators, besides the ones given, that lead to integrands amenable to this trick. But I was pleased with what I came up with. The question leads you through the opening paragraph and the extra parts depend very much on your having understood why the opening paragraph works. The reason for restriction  $0 \leq a < 3\pi/4$  is that the denominator should not be 0 for any value of  $x$  in the range of integration — otherwise, the integral is undefined. Writing  $\sin x + \cos x = \sqrt{2} \sin(x + \pi/4)$  shows that the denominator is first zero when  $x = 3\pi/4$ .

You might like to think how you would evaluate these integrals without using the trick method of this question. Perhaps the easiest way is to use the substitution  $t = \tan(\frac{1}{2}x)$  which converts the denominator to a quadratic in  $x$ .