

## Problem 23: Intersection of ellipses

(✓)

Show that the equation of any circle passing through the points of intersection of the ellipse

$$(x + 2)^2 + 2y^2 = 18$$

and the ellipse

$$9(x - 1)^2 + 16y^2 = 25$$

can be written in the form

$$x^2 - 2ax + y^2 = 5 - 4a .$$

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## Comments

You don't need to know anything about ellipses to do this question.

When this question was set it seemed too easy. But quite a high proportion of the candidates made no real progress. Of course, it is not easy to keep a cool head under examination conditions; but surely it is obvious that either  $x$  or  $y$  has to be eliminated from the two equations for the ellipses; and having decided that it is obvious which one to eliminate.

It is worth thinking about how many points of intersection of the ellipses we are expecting: a sketch might help if you know what shapes the ellipses are.

## Solution to problem 23

First we have to find the intersections of the two ellipses by solving the simultaneous equations

$$(x + 2)^2 + 2y^2 = 18 \quad (1)$$

$$9(x - 1)^2 + 16y^2 = 25 \quad (2)$$

We can eliminate  $y$  by multiplying equation (1) by 8 and subtracting equation (2), so that at the intersections

$$8(x + 2)^2 - 9(x - 1)^2 = 144 - 25$$

i.e.

$$x^2 - 50x + 96 = 0$$

i.e.

$$(x - 2)(x - 48) = 0 .$$

The two possible values for  $x$  at the intersections are therefore 2 and 48.

Next we find the values of  $y$  at the intersection. Taking  $x = 2$  and substituting into equation (1) gives  $16 + 2y^2 = 18$ , so  $y = \pm 1$ . Taking  $x = 48$  gives  $50^2 + 2y^2 = 18$  which has no (real) roots. Thus there are two points of intersection, at  $(2, 1)$  and  $(2, -1)$ .

Now we go after the circle. Suppose that a circle through the points of intersection has centre  $(p, q)$  and radius  $R$ . Then the equation of the circle is

$$(x - p)^2 + (y - q)^2 = R^2 .$$

Setting  $(x, y) = (2, 1)$  and  $(x, y) = (2, -1)$  gives two equations:

$$(2 - p)^2 + (1 - q)^2 = R^2 , \quad (2 - p)^2 + (-1 - q)^2 = R^2 .$$

Subtracting the two equations gives  $q = 0$  (it is obvious anyway, because of the symmetry of both ellipses under reflections in the  $x$  axis, that the centre of the circle must lie on the  $y$  axis). Thus the equation of any circle passing through the intersections is

$$(x - p)^2 + y^2 = (2 - p)^2 + 1 ,$$

which simplifies to the given result with  $p = a$ .

## Post-mortem

As commented earlier, there really wasn't much to this question apart from solving simultaneous equations and quadratic equations. I suppose that the daunting feature is that very little help is given in the way of intermediate steps of answers.

Another daunting feature is the unexplained parameter  $a$  in the equation of the circle. Two ellipses can intersect in four points, three points (if the ellipses touch rather than intersect at one of the points), two points, one point or not at all.

In general, we wouldn't expect to be able to draw a circle through four given points. We would expect exactly one circle through three given points (not lying on a line), but a whole family of circles through any two points. Our ellipses intersect in two points, which is why the circle depends on a parameter  $a$ .