Problem 22: Gregory's series (✓✓)

Give rough sketches of the function $\tan^k\theta$ for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$ in the two cases $k=1$ and $k \gg 1$.

(i) Show that for any positive integer n

$$\int_0^{\frac{1}{4}\pi} \tan^{2n+1}\theta \,\mathrm{d}\theta = (-1)^n\left(\tfrac{1}{2}\ln 2 + \sum_{m=1}^{n}\frac{(-1)^m}{2m}\right), \tag{$\dagger$}$$

and deduce that

$$\ln 2 = -\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\,. \tag{$\ddagger$}$$

(ii) Show similarly that

$$\frac{\pi}{4} = -\sum_{m=1}^{\infty}\frac{(-1)^m}{2m-1}\,.$$

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Comments

The symbol $\gg$ in the first paragraph means 'much greater than', so for the second sketch k is a large number.

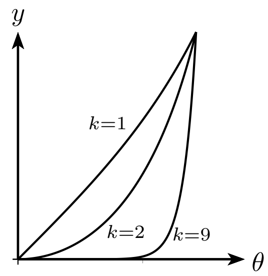
This is a good question. In part (i) you are told what to do (in not-very-easy stages) and part (ii) tests your understanding of what you have done and why you have done it by asking you to apply the method to a different but essentially similar problem.

In the first paragraph, you have to see how the function $\tan^k\theta$ changes when k increases. You need only a rough sketch to show that you have understood the important point. This should be done by thought, not by means of a calculator.

If you are stuck with the integral of the second paragraph, you might like to think in terms of a recurrence formula, i.e. a formula relating I_{2n+1} and I_{2n-1} (in the obvious notation).

The series derived in part (ii) for $\frac{1}{4}\pi$ is usually called Leibniz's formula, although the general series for $\tan^{-1}x$ was written down by Gregory in 1671, two years before Leibniz. It was one of the first explicit formulae for π, though Wallis had obtained a product formula in 1655 using a method similar to the method of this question, using $\sin^k x$ in the integral. Previously, the value of π could only be estimated geometrically, by (for example) approximating the circumference of a circle by the edges of an inscribed regular polygon. Using a square gives $\pi \approx 2\sqrt{2}$.

Solution to problem 22



For $0 \leqslant \tan\theta < \frac{1}{4}\pi$, we have $\tan\theta < 1$, so that the curve $y = \tan^k\theta$ is close to zero (i.e. much smaller than 1) when k is large. This is illustrated in the figure which shows three cases: for $k = 1$ the graph is mildly curved; for larger k the graph hugs the x-axis before taking off. The graphs all pass through the point $(\frac{1}{4}\pi, 1)$.

(i) To evaluate the integral, let

$$I_{2n+1} = \int_0^{\frac{1}{4}\pi} \tan^{2n+1}\theta \ \mathrm{d}\theta\,. \qquad (**)$$

We shall express I_{2n+1} in terms of I_{2n-1} using the relation $\tan^2\theta = \sec^2\theta - 1$:

$$I_{2n+1} = \int_0^{\frac{1}{4}\pi} \tan^{2n-1}\theta\,(-1+\sec^2\theta)\,\mathrm{d}\theta = -I_{2n-1} + \int_0^1 u^{2n-1}\,\mathrm{d}u = -I_{2n-1} + \frac{1}{2n}\,.$$

To evaluate the second integral, set $u = \tan\theta$ so that $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$.

Repeating the process gives

$$I_{2n+1} = -I_{2n-1} + \frac{1}{2n} = I_{2n-3} - \frac{1}{2(n-1)} + \frac{1}{2n} = \cdots = (-1)^n I_1 + \frac{1}{2n} - \frac{1}{2(n-1)} + \cdots + (-1)^{n+1}\frac{1}{2}\,.$$

The above sum (starting with $1/(2n)$) is the same as that in (†) overleaf, so it only remains to evaluate I_1, corresponding to $n = 0$ in $(**)$:

$$I_1 = \int_0^{\frac{1}{4}\pi} \tan\theta\ \mathrm{d}\theta = -\ln(\cos\theta)\Big|_0^{\frac{1}{4}\pi} = -\ln(1/\sqrt{2}) = \tfrac{1}{2}\ln 2\,,$$

as required.

We deduce the expression (‡) overleaf for $\ln 2$ using the first line of the question. When n is very large, I_{2n+1} is very small, being the area under a graph which is almost zero for almost all of the range of integration. In the limit $n \to \infty$, we set $I_{2n+1} = 0$ in (†) which leads immediately to (‡).

(ii) To obtain the formula for $\frac{1}{4}\pi$, we follow the above method using I_{2n} instead of I_{2n+1}. This time we have to calculate I_0:

$$I_0 = \int_0^{\frac{1}{4}\pi} 1\ \mathrm{d}\theta = \frac{\pi}{4}\,.$$

Post-mortem

You might think that the method of obtaining the formula for $\frac{1}{4}\pi$ in this question is rather indirect; one could instead just integrate the formula

$$\frac{\mathrm{d}\tan^{-1}x}{\mathrm{d}x} = \frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots \qquad (*)$$

term by term and get the result immediately by setting $x = 1$. The virtue of the method used in the question is that it gives an explicit form (an integral) of the remainder after n terms of the series. We were able to show, by means of a sketch, that the remainder tends to zero as n tends to infinity; in other words, we showed that the series converges. Although the sketch method of proof is a bit crude, it can easily be made more rigorous once the concept of integration is more carefully defined. On the other hand, integrating $(*)$ and setting $x = 1$ is a bit delicate, since the series only converges for $x^2 < 1$.