

Problem 21: Melting snowballs

(✓✓)

Frosty the snowman is made from two uniform spherical snowballs, initially of radii $2R$ and $3R$. The smaller (which is his head) stands on top of the larger. As each snowball melts, its volume decreases at a rate which is directly proportional to its surface area, the constant of proportionality being the same for each snowball. During melting, the snowballs remain spherical and uniform. When Frosty is half his initial height, show that the ratio of his volume to his initial volume is $37 : 224$.

What is this ratio when Frosty is one tenth of his initial height?

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Comments

To start with, you have to set up a differential equation which gives the radii of the snowballs as a function of time. Don't worry if you have never solved a differential equation: you will be able to solve this one. Having solved it, you have to evaluate the constant of integration from the information in the question.

Setting up and solving the differential equation is not difficult in itself, but it is necessary to think what variables you want to use and then go through a number of steps in the dark, without any reassurance from the question.

Solution to problem 21

For either snowball, the rate of change of volume, call it v , at any time is related to the surface area a by

$$\frac{dv}{dt} = -ka, \quad (*)$$

where k is a positive constant. For a sphere of radius r , this becomes

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -k(4\pi r^2).$$

We can write $\frac{d(r^3)}{dt}$ as $3r^2\frac{dr}{dt}$, so equation (*) is equivalent to (cancelling the factor of $4\pi r^2$)

$$\frac{dr}{dt} = -k.$$

Thus $r = -kt + C$, where C is a constant of integration.

Initially, Frosty's head has radius $2R$ and his body has radius $3R$, so the equations for the radii of the head and body at time t are respectively

$$r = -kt + 2R \quad \text{and} \quad r = -kt + 3R.$$

Frosty's height h is twice the sum of these radii, i.e. $h = 2(-kt + 5R)$, which falls to half its original value of $10R$ when $kt = \frac{5}{4}R$. At this time, the radii of the head and body are $\frac{3}{4}R$ and $\frac{7}{4}R$, so the ratio of his volume to his initial volume is

$$\frac{\left(\frac{4}{3}\pi\right)\left(\frac{3}{4}R\right)^3 + \left(\frac{4}{3}\pi\right)\left(\frac{7}{4}R\right)^3}{\left(\frac{4}{3}\pi\right)(2R)^3 + \left(\frac{4}{3}\pi\right)(3R)^3} = \frac{\left(\frac{3}{4}\right)^3 + \left(\frac{7}{4}\right)^3}{2^3 + 3^3} = \frac{37}{224}.$$

When Frosty is just R high, all that remains of him, since $kt > 2R$, is the body, which is a sphere of radius $\frac{1}{2}R$. The ratio of his volume to his original volume is just $\frac{1}{240}$.

Post-mortem

As mentioned in the comments section, there is not a lot for you to do in this question, but it is a lot for you to do entirely on your own.

The key is to transcribe the words in the question into mathematics, the obvious starting point being equation (*).

The original question had a different rider¹⁸, asking about the maximum rate of change of volume with respect to area. I didn't much like this, because it seemed to be a new and not very interesting idea. My rider relates directly to the previous part, and requires a tiny bit of extra thought (a trap, some would say, but it is only a trap if you are on autopilot).

¹⁸ The term *rider* seems a bit old-fashioned now. It referred to the final part at the end of an examination question in the days when examination questions often consisted of a relatively straightforward first part, perhaps the proof of a theorem, followed by a more tricky part extending or applying the first part.