

Problem 19: Logarithms

(✓✓✓)

To nine decimal places, $\log_{10} 2 = 0.301029996$ and $\log_{10} 3 = 0.477121255$.

- (i) Calculate $\log_{10} 5$ and $\log_{10} 6$ to three decimal places. By taking logs, or otherwise, show that

$$5 \times 10^{47} < 3^{100} < 6 \times 10^{47}.$$

Hence write down the first digit of 3^{100} .

- (ii) Find the first digit of each of the following numbers: 2^{1000} ; $2^{10\,000}$; and $2^{100\,000}$.

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Comments

This nice little question shows why it is a good idea to ban calculators from some mathematics examinations — though I notice that my calculator can't work out 2^{10000} .

When I was at school, before electronic calculators were invented, we had to spend quite a lot of time in year 8 (I think) doing extremely tedious calculations by logarithms. We were provided with a book of tables of four-figure logarithms and lots of uninteresting numbers to multiply or divide. The book also had tables of trigonometric functions. When it came to antilogging, to get the answer, we had to use the tables backwards, since no tables of inverse logarithms (exponentials) were provided.

A logarithm (to base 10, as always in this question) consists of two parts: the *characteristic* which is the number before the decimal point and the *mantissa* which is the number after the decimal point. It is the mantissa that gives the significant figures of the number that has been logged; the characteristic tells you where to put the decimal point. The important property of logs to base 10 is that $A \times 10^n$ and $A \times 10^m$ have the same mantissa, so log tables need only show the mantissa.

The characteristic of a number greater than 1 is non-negative but the characteristic of a number less than 1 is negative. The rules for what to do in the case of a negative characteristic were rather complicated: you couldn't do ordinary arithmetic because the logarithm consisted of a negative characteristic and a positive mantissa. In ordinary arithmetic, the number -3.4 means $-3 - 0.4$ whereas the corresponding situation in logarithms, normally written $\bar{3}.4$, means $-3 + 0.4$. Instead of explaining this, the teacher gave a complicated set of rules, which just had to be learned — not the right way to do mathematics.

This question is all about calculating mantissas and there are no negative characteristics, I'm happy to say.

Solution to problem 19

(i) For the very first part, we have

$$\log 2 + \log 5 = \log 10 = 1, \text{ so } \log 5 = 1 - 0.301029996 = 0.699 \text{ (3 d.p.)}$$

and

$$\log 6 = \log 2 + \log 3 = 0.778 \text{ (3 d.p.)}.$$

Now we have to show that

$$5 \times 10^{47} < 3^{100} < 6 \times 10^{47}. \quad (*)$$

Taking logs preserves the inequalities (because $\log x$ is an *increasing* function), so we need to show that

$$47 + \log 5 < 100 \log 3 < 47 + \log 6$$

i.e. that

$$47 + 0.699 < 47.7121 < 47 + 0.778$$

which is true. We see from (*) that the first digit of 3^{100} is 5.

(ii) To find the first digit of these numbers, we use the method of part (i).

We have (to 3 d.p.)

$$\log 2^{1000} = 1000 \log 2 = 301.030 = 301 + 0.030 < 301 + \log 2$$

Thus $10^{301} < 2^{1000} < 10^{301} \times 2$ and the first digit of 2^{1000} is 1.

Similarly,

$$\log 2^{10\,000} = 10\,000 \log 2 = 3010.29996 < 3010 + \log 2, \text{ so the first digit of } 2^{10\,000} \text{ is 1.}$$

Finally,

$$\log 2^{100\,000} = 30102 + 0.9996 \text{ (4 d.p.) and}$$

$\log 9 = 2 \log 3 = 0.95 \text{ (2 d.p.)}$, so the first digit of $2^{100\,000}$ is 9.

Post-mortem

Although the ideas in this question are really quite elementary, you needed to understand them deeply. You should feel pleased with yourself if you get this one out.

Note that in part (i) I started with (*), the result I was trying to prove and then showed it is true. This is of course dangerous. But provided you keep writing ‘We have to prove that ...’ or ‘RTP’ (Required To Prove) you should not get muddled between what you have proved and what you are trying to prove.