

Problem 16: Non-linear simultaneous equations

(✓)

Consider the system of equations

$$2yz + zx - 5xy = 2$$

$$yz - zx + 2xy = 1$$

$$yz - 2zx + 6xy = 3.$$

Show that

$$xyz = \pm 6$$

and find the possible values of x , y and z .

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Comments

At first sight, this looks forbidding. A closer look reveals that the variables x , y and z occur only in pairs yz , zx and xy . The problem therefore boils down to solving three simultaneous equations in the variables yz , zx and xy , then using the solution to find x , y and z individually.

What do you make of the equation $xyz = \pm 6$? How could the $\pm$ arise?

There are two ways of tackling such simultaneous equations. You could use the first equation to find an expression for one variable (yz say) in terms of the other two variables, then substitute this into the other equations to eliminate yz from the system. Then use the second equation (in its new form) to find an expression for one of the two remaining variables (zx say), and substitute this into the third equation (in its new form) to obtain an equation for the third variable (xy). Having solved this equation, you can substitute back to find the other variables. This method is called *Gauss elimination*.

Alternatively, you could eliminate one variable (yz say) from the first two of equations by multiplying the first equation by something suitable and the second equation by something suitable and subtracting. You then eliminate yz from the second and third equations similarly. That leaves you with two simultaneous equations in two variables which you can solve by your favourite method.

There is another way of solving the simultaneous equations, which is better in theory than in practice. You write the equations in matrix form $\mathbf{M}\mathbf{x} = \mathbf{c}$, where in this case

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & -5 \\ 1 & -1 & 2 \\ 1 & -2 & 6 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

The solution is then $\mathbf{x} = \mathbf{M}^{-1}\mathbf{c}$. All you have to do is invert a 3×3 matrix, which is doable but not very pleasant. You might like to try it if you know the formula for the inverse of a matrix.

Solution to problem 16

Start by labelling the equations:

$$2yz + zx - 5xy = 2, \quad (1)$$

$$yz - zx + 2xy = 1, \quad (2)$$

$$yz - 2zx + 6xy = 3. \quad (3)$$

We use Gaussian elimination. Rearranging equation (1) gives

$$yz = -\frac{1}{2}zx + \frac{5}{2}xy + 1, \quad (4)$$

which we substitute back into equations (2) and (3) :

$$-\frac{3}{2}zx + \frac{9}{2}xy = 0, \quad (5)$$

$$-\frac{5}{2}zx + \frac{17}{2}xy = 2. \quad (6)$$

Thus $zx = 3xy$ (using equation (5)). Substituting into equation (6) gives $xy = 2$ and $zx = 6$. Finally, substituting back into equation (1) shows that $yz = 3$.

The question is now plain sailing. Multiplying the three values together gives $(xyz)^2 = 36$ and taking the square root gives $xyz = \pm 6$ as required.

Now it remains to solve for x , y and z individually. We know that $yz = 3$, so if $xyz = +6$ then $x = +2$, and if $xyz = -6$ then $x = -2$. The solutions are therefore either $x = +2$, $y = 1$, and $z = 3$ or $x = -2$, $y = -1$, and $z = -3$.

Post-mortem

There were two key observations which allowed us to do this question quite easily. Both came from looking carefully at the question. The first was that the given equations, although non-linear in x , y and z (they are quadratic, since they involve products of these variables) could be thought of as three linear equations in yz , zx and xy . That allowed us to make a start on the question. The second observation was that the equation $xyz = \pm 6$ is almost certain to come from $(xyz)^2 = 36$ and that gave us the next step after solving the simultaneous equations. (Recall the next step was to multiply all the variables together.)

There was a point of technique in the solution: it is often very helpful in this sort of problem (and many others) to number your equations. This allows you to refer back clearly and quickly, for your benefit as well as for the benefit of your readers.

The three simultaneous quadratic equations (1) – (3) have a geometric interpretation but it is not at all obvious. The equations are quadratic in the variables x , y and z , which means that each equation represents either an ellipsoid or a hyperboloid (or some special cases).¹⁶ The solutions of all three equations represent the points of intersection of the three surfaces. Not very easy to picture.

¹⁶ An *ellipsoid* is roughly the shape of the surface of a rugby ball, or of the giant galaxy ESO 325-G004. A *hyperboloid* can either be the shape of an infinite radar dish (in fact, a pair of such dishes) or it can be the shape of a power station cooling tower. Our equations in fact represent hyperboloids of the cooling-tower type.