

Problem 13: Binomial expansion

(✓)

- (i) Use the first four terms of the binomial expansion of $(1 - \frac{1}{50})^{\frac{1}{2}}$ to derive the approximation $\sqrt{2} \approx 1.414214$.
- (ii) Calculate similarly an approximation to the cube root of 2 to six decimal places by considering $(1 + \frac{N}{125})^{\frac{1}{3}}$, where N is a suitable number.

[You need not justify the accuracy of your approximations.]

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Comments

Although you do not have to justify your approximations, you do need to think carefully about the number of terms required in the expansions. You first have to decide how many you will need to obtain the given number of decimal places; then you have to think about whether the next term is likely to affect the value of the last decimal.

It is not at all obvious where the $\sqrt{2}$ in part (i) comes from until you write $(1 - \frac{1}{50})^{\frac{1}{2}} = \sqrt{\frac{49}{50}}$.

For part (ii), you have to choose N in such a way that $125 + N$ has something to do with a power of 2.

You can make the arithmetic of the binomial expansions a bit easier by arranging the denominator to be a power of 10 so that, for example, the expansion in part (i) becomes $(1 - \frac{2}{100})^{\frac{1}{2}}$. No need to use a calculator.

Solution to problem 13

(i) First we expand binomially:

$$\begin{aligned} & \left(1 - \frac{2}{100}\right)^{\frac{1}{2}} \\ &= 1 + \left(\frac{1}{2}\right)\left(-\frac{2}{100}\right) + \left(\frac{1}{2!}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{2}{100}\right)^2 - \left(\frac{1}{3!}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{2}{100}\right)^3 + \dots \\ &\approx 1 - \frac{1}{100} - \frac{5}{10^5} - \frac{5}{10^7} = 0.9899495. \end{aligned}$$

It is clear that the next term in the expansion would introduce the eighth decimal place, which it seems we do not need. Of course, after further manipulations we might find that the above calculation does not supply the 6 decimal places we need for $\sqrt{2}$, in which case we will have to work out the next term in the expansion.

$$\text{But } \left(\frac{98}{100}\right)^{\frac{1}{2}} = \frac{7\sqrt{2}}{10}, \text{ so } \sqrt{2} \approx 9.899495/7 \approx 1.414214.$$

(ii) We have

$$\begin{aligned} & \left(1 + \frac{3}{125}\right)^{\frac{1}{3}} \\ &= 1 + \left(\frac{1}{3}\right)\left(\frac{3}{125}\right) + \left(\frac{1}{2!}\right)\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(\frac{3}{125}\right)^2 + \left(\frac{1}{3!}\right)\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(\frac{3}{125}\right)^3 + \dots \\ &\approx 1 + \frac{8}{1000} - \frac{64}{10^6} + \frac{5}{3} \frac{8^3}{10^9} \\ &= 1.007936 + \frac{256}{3} \frac{1}{10^8} = 1.007937. \end{aligned}$$

Successive terms in the expansion decrease by a factor of about 1000, so this should give the right number of decimal places.

$$\text{But } \left(\frac{128}{125}\right)^{\frac{1}{3}} = \frac{4\sqrt[3]{2}}{5} = \frac{8\sqrt[3]{2}}{10} \text{ so } \sqrt[3]{2} \approx \frac{10.0793}{8} \approx 1.259921.$$

Post-mortem

This question required a bit of intuition, and some accurate arithmetic. You don't have to be brilliant at arithmetic to be a good mathematician, but most mathematicians aren't bad at it. There have in the past been children who were able to perform extraordinary feats of arithmetic. For example, Zerah Colburn, a 19th-century American, toured Europe at the age of 8. He was able to multiply instantly any two 4-digit numbers given to him by the audiences. George Parker Bidder (the Calculating Boy) could perform similar feats, though unlike Colburn he became a distinguished mathematician and scientist. One of his brothers knew the bible by heart.

The error in the approximation is the weak point of this question. Although it is clear that the next term in the expansion is too small to affect the accuracy, it is not obvious that the sum of all the next hundred (say) terms of the expansion is negligible (though in fact it is). What is needed is an estimate of the truncation error in the binomial expansion. Such an estimate is not hard to obtain (first-year university work) and is typically of the same order of magnitude as the first neglected term in the expansion. Without this estimate, the approximation is not justified.