

Problem 12: Maximising with constraints

(✓)

Prove that the rectangle of greatest perimeter which can be inscribed in a given circle is a square. The result changes if, instead of maximising the sum of lengths of sides of the rectangle, we seek to maximise the sum of n th powers of the lengths of those sides for $n \geq 2$. What happens if $n = 2$? What happens if $n = 3$? Justify your answers.

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Comments

Obviously, the perimeter of a *general* rectangle has no maximum (it can be as long as you like), so the key to this question is to use the constraint that the rectangle lies in the circle.

The vertices of a rectangle of greatest perimeter must lie *on* the circle; this seems obvious, but you should devote the first sentence of your solution to justifying it. And draw a diagram (I didn't because there wasn't enough room on the page).

The word 'greatest' in the question immediately suggests differentiating something. The perimeter of a rectangle is expressed in terms of two variables, length x and breadth y , so you must find a way of using the constraint to eliminate one variable. This could be done in two ways: express y in terms of x , or express both x and y in terms of another variable (an angle, say).

Finally, there is the matter of deciding whether your solution is the greatest or least value (or neither). This can be done either by considering second derivatives, or by trying to understand the different situations. Often the latter method, or a combination, is preferable.

The theory of stationary values of functions subject to constraints is very important: it has widespread applications (for example, in theoretical physics, financial mathematics and in fact in almost any area to which mathematics is applied). It forms a whole branch of mathematics called optimisation. Normally, the methods described above (eliminating one variable) are not used: a clever idea, the method of Lagrange multipliers, is used instead.

Solution to problem 12

The vertices of the rectangle must lie on the circle. Suppose not. Then two adjacent vertices (at least) lie in the interior of the circle. We could then increase the perimeter by extending two sides beyond these vertices. [You should include a diagram here!]

Let the circle have diameter d and let the length of one side of the rectangle be x and the length of the adjacent side be y .

Then, by Pythagoras's theorem,

$$y = \sqrt{d^2 - x^2} \quad (\dagger)$$

and the perimeter P is given by

$$P = 2x + 2\sqrt{d^2 - x^2}.$$

We can find the largest possible value of P as x varies by calculus. We have

$$\frac{dP}{dx} = 2 - 2\frac{x}{\sqrt{d^2 - x^2}},$$

so for a stationary point we require (cancelling the factor of 2 and squaring)

$$1 = \frac{x^2}{d^2 - x^2} \quad \text{i.e.} \quad 2x^2 = d^2.$$

Thus $x = d/\sqrt{2}$ (ignoring $x = -d/\sqrt{2}$ for obvious reasons). Substituting this into $(\dagger)$ gives $y = d/\sqrt{2}$, so the rectangle is indeed a square, with perimeter $2\sqrt{2}d$.

But is this the maximum perimeter? The easiest way to investigate is to calculate the second derivative, which is easily seen to be negative for all values of x , and in particular when $x = d/\sqrt{2}$. The stationary point is certainly a maximum. Alternatively, we could argue that the rectangle for which $x = 0$ has perimeter $2d$ which is less than $2\sqrt{2}d$ so $x = d/\sqrt{2}$ cannot correspond to the minimum perimeter.

For the second part, we consider

$$f(x) = x^n + (d^2 - x^2)^{\frac{n}{2}}.$$

The first thing to notice is that f is constant if $n = 2$, so in this case, the largest (and smallest) value is d^2 . You can use Pythagoras's theorem to see why this result holds.

For $n = 3$, we have

$$f'(x) = 3x^2 - 3x(d^2 - x^2)^{\frac{1}{2}},$$

so $f(x)$ is stationary when $x^4 = x^2(d^2 - x^2)$, i.e. when $2x^2 = d^2$ as before or when $x = 0$. The corresponding stationary values of f are $\sqrt{2}d^3$ and $2d^3$, so this time the largest value occurs when $x = 0$.

Post-mortem

You will almost certainly want to investigate the situation for other values of n yourself, just to see what happens.

Were you happy with the proof given above that the square has the *largest* perimeter? And isn't it a bit odd that we did not discover a stationary point corresponding to the smallest value (which can easily be seen to occur at $x = 0$ or $x = d$)? To convince yourself that the maximum point gives the largest value, it is a good idea to sketch a graph: you can check that dP/dx is positive for $0 < x < d/\sqrt{2}$ and negative for $d/\sqrt{2} < x < d$. The smallest values of the perimeter occur at the endpoints of the interval $0 \leq x \leq d$ and therefore do not have to be turning points.

A much better way of tackling the problem is to set $x = d \cos \theta$ and $y = d \sin \theta$ and find the perimeter as a function of θ . Because θ can take any value (there are no end points such as $x = 0$ to consider), the largest and smallest perimeters correspond to stationary points. Try it.