Problem 11: Egyptian fractions (✓)

A number of the form $\frac{1}{N}$, where N is an integer greater than 1, is called a *unit fraction*. Noting that

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6} \text{ and } \frac{1}{3} = \frac{1}{4} + \frac{1}{12},$$

guess a general result of the form

$$\frac{1}{N} = \frac{1}{a} + \frac{1}{b} \qquad (*)$$

and hence prove that any unit fraction can be expressed as the sum of two distinct unit fractions. By writing $(*)$ in the form

$$(a - N)(b - N) = N^2$$

and by considering the factors of N^2, show that if N is prime, then there is only one way of expressing $\frac{1}{N}$ as the sum of two distinct unit fractions.

Prove similarly that any fraction of the form $\frac{2}{N}$, where N is prime number greater than 2, can be expressed uniquely as the sum of two distinct unit fractions.

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Comments

Fractions written as the sum of unit fractions are called *Egyptian fractions*: they were used by Egyptians. The earliest record of such use is 1900BC. The Rhind papyrus in the British Museum gives a table of representations of fractions of the form $\frac{2}{n}$ as sums of unit fractions for all odd integers n between 5 and 101 — a remarkable achievement when you consider that algebra was 3,500 years in the future.

It is not clear why Egyptians represented fractions this way; maybe it just seemed a good idea at the time. Certainly the notation they used, in which for example $\frac{1}{n}$ was denoted by n with an oval on top, does not lend itself to generalisation to fractions that are not unit. One of the rules for expressing non-unit fractions in terms of unit fractions was that all the unit fractions in should be distinct, so $\frac{2}{7}$ had to be expressed as $\frac{1}{4} + \frac{1}{28}$ instead of as $\frac{1}{7} + \frac{1}{7}$, which seems pretty daft.

It is not obvious that every fraction can be express in Egyptian form; this was proved by Fibonacci in 1202. There are however still many unsolved problems relating to Egyptian fractions.

Egyptian fractions have been called a wrong turn in the history of mathematics; if so, it was a wrong turn that favoured style over utility; no bad thing, in my opinion.

Solution to problem 11

A good guess would be that the first term of the decomposition of $\dfrac{1}{N}$ is $\dfrac{1}{N+1}$, i.e. $a = N+1$. In that case, the other term is $\dfrac{1}{N} - \dfrac{1}{N+1}$ i.e. $b = N(N+1)$. That proves the result that every unit fraction can be expressed as the sum of two unit fractions.

The only factors of N^2 (since N is prime) are 1, N and N^2. The possible factorisations of N^2 are therefore $N^2 = 1 \times N^2$ or $N^2 = N \times N$. We discount $N \times N$, since this would lead to $a = b$, and this is ruled out in the question. That leaves only $a - N = 1$ and $b - N = N^2$ (or the other way round). Thus $N+1$ and $N^2 + N$ are the only possible values for a and b and the decomposition is unique.

For the second half, set

$$\frac{2}{N} = \frac{1}{a} + \frac{1}{b}$$

where $a \neq b$. We need to aim for an equation with N^2 on one side, so that we can use the method of the first part. We have $ab - \frac{1}{2}(a+b)N = 0$ which we write as

$$\left(a - \frac{N}{2}\right)\left(b - \frac{N}{2}\right) = \frac{N^2}{4} \quad \text{i.e.} \quad (2a - N)\,(2b - N) = N^2 \,.$$

Thus $2a - N = N^2$ and $2b - N = 1$ (or the other way round). The decomposition is therefore unique and given by

$$\frac{2}{N} = \frac{1}{\frac{1}{2}(N^2 + N)} + \frac{1}{\frac{1}{2}(N+1)} \,.$$

The only possible fly in the ointment is the $\frac{1}{2}$ in the denominators: a and b are supposed to be integers. However, all prime numbers greater than 2 are odd, so $N+1$ and $N^2 + N$ are both even and the denominators are indeed integers.

Post-mortem

Another way of getting the last part (less systematically) would have been to notice that

$$\frac{1}{N} = \frac{1}{(N+1)} + \frac{1}{N(N+1)} \implies \frac{2}{N} = \frac{1}{\frac{1}{2}(N+1)} + \frac{1}{\frac{1}{2}N(N+1)}$$

which gives the result immediately. How might you have noticed this? Well, I noticed it by trying to work out some examples, starting with what is given at the very beginning of the question. If $\dfrac{1}{3} = \dfrac{1}{4} + \dfrac{1}{12}$ then $\dfrac{2}{3} = \dfrac{2}{4} + \dfrac{2}{12}$ which works.

Of course, the advantage of the systematic approach is that it allows for generalisation: what happens if N is odd but not prime; what happens if the numerator is 3 not 2?