

Problem 10: True or false

(✓✓)

Which of the following statements are true and which are false? Justify your answers.

- (i) $a^{\ln b} = b^{\ln a}$ for all positive numbers a and b .
- (ii) $\cos(\sin \theta) = \sin(\cos \theta)$ for all real θ .
- (iii) There exists a polynomial P such that $|P(x) - \cos x| \leq 10^{-6}$ for all (real) x .
- (iv) $x^4 + 3 + x^{-4} \geq 5$ for all $x > 0$.

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Comments

The four parts are (somewhat annoyingly) related only by the fact that you have to decide whether each statement is true or false.

If true, the justification has to be a proof. If false, you could prove that it is false in some general way, but it is nearly always better to find a simple counterexample — as simple as possible.

Part (iii) might look a bit odd. I suppose it relates to the standard approximation $\cos x \approx 1 - \frac{1}{2}x^2$ which holds when x is small. This result can be improved by using a polynomial of higher degree: the next term is $+\frac{1}{4!}x^4$. It can be proved that $\cos x$ can be approximated as accurately as you like for small x by a polynomial of the form

$$\sum_{n=0}^N (-1)^n \frac{x^{2n}}{(2n)!}$$

(the truncated Maclaurin expansion). You have to use more terms of the approximation (i.e. a larger value of N) if you want either greater accuracy or larger x . In part (iii) of this question, you are being asked if there is a polynomial such that the approximation is good for all values of x .

I've awarded the question ✓✓ for difficulty, just because quite a few good ideas are required.

Solution to problem 10

(i) True. The easiest way to see this is to log both sides. For the left hand side, we have

$$\ln(a^{\ln b}) = (\ln b)(\ln a)$$

and for the right hand side we have

$$\ln(b^{\ln a}) = (\ln a)(\ln b),$$

which agree.

Note that we have to be a bit careful with this sort of argument. The argument used is that $A = B$ because $\ln A = \ln B$. This requires the property of the $\ln$ function that $\ln A = \ln B \Rightarrow A = B$. You can easily see that this property holds because $\ln$ is a strictly increasing function; if $A > B$, then $\ln A > \ln B$. The same would not hold for (say) $\sin$ (i.e. $\sin A = \sin B \not\Rightarrow A = B$).

(ii) False. $\theta = \frac{1}{2}\pi$ is an easy counterexample. Even though it is 'obvious' we still need to show that $\cos 1 \neq 0$, which we could do by noting that $0 < 1 < \frac{1}{3}\pi$ and sketching the graph of $\cos x$.

(iii) False. Roughly speaking, any polynomial can be made as large as you like by taking x to be very large (provided it is of degree greater than zero), whereas $|\cos x| \leq 1$; and there is obviously no polynomial of degree zero (i.e. no constant number) for which the statement holds.

But how can we write this out formally? First let us knock off the case when the polynomial is of degree zero, i.e. a constant, call it P . Then either $P \geq \frac{1}{2}$ or $P < \frac{1}{2}$. In either case, P cannot be close to both $\cos 0$ and $\cos \frac{1}{2}\pi$.

Now suppose

$$P(x) = a_N x^N + \sum_{n=0}^{N-1} a_n x^n \quad (*)$$

where $N \geq 1$ and assume $a_N > 0$. It is enough to show that $P(x) > 2$ for some value of x . We can find a number x so large that $a_N x^N > N|a_n|x^n + 2$ for each integer n with $0 \leq n \leq N-1$. The smallest possible value of $P(x)$ for any given positive x would be achieved if all the coefficients in the sum were negative. Thus

$$P(x) \geq a_N x^N - \sum_{n=0}^{N-1} |a_n| x^n > a_N x^N + (2 - a_N x^N) = 2$$

and we are done.

(iv) True: $x^4 + 3 + x^{-4} = (x^2 - x^{-2})^2 + 5 \geq 5$.

Post-mortem

The important point here is that if you want to show a statement is true, you have to give a formal proof, whereas if you want to show that it is false, you only need give one counterexample. It does not have to be an elaborate counterexample — in fact, the simpler the better.

My proof for part (iii) is more elaborate than could have been expected of candidates in the examination. A sketch of a polynomial of degree N would have been enough, provided the special case of $N = 0$ was dealt with separately.