

Problem 9: Integration by substitution

(✓)

Show, by means of a change of variable or otherwise, that

$$\int_0^\infty f((x^2 + 1)^{\frac{1}{2}} + x) dx = \frac{1}{2} \int_1^\infty (1 + t^{-2})f(t) dt ,$$

for any given function f .

Hence, or otherwise, show that

$$\int_0^\infty ((x^2 + 1)^{\frac{1}{2}} + x)^{-3} dx = \frac{3}{8} .$$

1998 Paper I

Comments

Note that ‘by change of variable’ means the same as ‘by substitution’.

There are two things to worry about when you are trying to find a change of variable to convert one integral to another: you need to make the integrands match up and you need to make the limits match up. Sometimes, the limits give the clue to the change of variable. (For example, if the limits on the original integral were 0 and 1 and the limits on the transformed integral were 0 and $\frac{1}{4}\pi$, then an obvious possibility would be to make the substitution $t = \tan x$). Here, the change of variable is determined by the integrand, since it must work for all choices of f .

Perhaps you are worried about the infinite upper limit of the integrals. If you are trying to prove some rigorous result about infinite integrals, you might use the definition

$$\int_0^\infty f(x) dx = \lim_{a \rightarrow \infty} \int_0^a f(x) dx ,$$

but for present purposes you just do the integral and put in the limits. The infinite limit will not normally present problems. For example,

$$\int_1^\infty (x^{-2} + e^{-x}) dx = (-x^{-1} - e^{-x}) \Big|_1^\infty = -\frac{1}{\infty} - e^{-\infty} + \frac{1}{1} + e^{-1} = 1 + e^{-1} .$$

Don't be afraid of writing $1/\infty = 0$. It is perfectly OK to use this as shorthand for $\lim_{x \rightarrow \infty} 1/x = 0$; but ∞/∞ , $\infty - \infty$ and $0/0$ are definitely not OK, because of their ambiguity.

Solution to problem 9

Clearly, to get the argument of f right, we must set

$$t = (x^2 + 1)^{\frac{1}{2}} + x.$$

We must check that the new limits are correct (if not, we are completely stuck). When $x = 0$, $t = 1$ as required. Also, $(x^2 + 1)^{\frac{1}{2}} \rightarrow \infty$ as $x \rightarrow \infty$, so $t \rightarrow \infty$ as $x \rightarrow \infty$. Thus the upper limit is still ∞ , again as required.

Using the standard method of changing the variable in an integral (basically the chain rule), the integral becomes

$$\int_1^\infty f(t) \frac{dx}{dt} dt$$

so the next task is to find $\frac{dx}{dt}$. This we can do in two ways: we find x in terms of t and differentiate it; or we could find $\frac{dt}{dx}$ and turn it upside down. The snag with the second method is that the answer will be in terms of x , so we will have to express x in terms of t anyway—in which case, we may as well use the first method.

We start by finding x in terms of t :

$$t = (x^2 + 1)^{\frac{1}{2}} + x \Rightarrow (t - x)^2 = (x^2 + 1) \Rightarrow x = \frac{t^2 - 1}{2t} = \frac{t}{2} - \frac{1}{2t}.$$

Note that for each value of t there is exactly one value of x ; we need this for the substitution to work.

Then we differentiate:

$$\frac{dx}{dt} = \frac{1}{2} + \frac{1}{2t^2} = \frac{1}{2}(1 + t^{-2})$$

which is exactly the factor that we require for the transformed integrand given in the question.

For the last part, we take $f(t) = t^{-3}$. Thus

$$\begin{aligned} \int_0^\infty ((x^2 + 1)^{\frac{1}{2}} + x)^{-3} dx &= \frac{1}{2} \int_1^\infty (t^{-3} + t^{-5}) dt \\ &= \frac{1}{2} \left(\frac{t^{-2}}{-2} + \frac{t^{-4}}{-4} \right) \Big|_1^\infty \\ &= \frac{1}{2} \left(\frac{1}{2} + \frac{1}{4} \right) = \frac{3}{8}, \text{ as required.} \end{aligned}$$

Post-mortem

This is not a difficult question conceptually once you realise the significance of the fact that the change of variable must work whatever function f is in the integrand.

There was a useful point connected with calculating $\frac{dx}{dt}$. It was a good idea not to plunge into the algebra without first thinking about alternative methods; in particular, we might have used (but didn't)

$$\frac{dx}{dt} = 1 / \frac{dt}{dx}.$$

Finally, there was the infinite limit of the integrand, which I hope you saw was not something to worry about (even though infinite limits are excluded from the A-level Mathematics specifications). If you were setting up a formal definition of what an integral is, you would have to use finite limits, but if you are merely calculating the value of an integral, you just go ahead and do it, with whatever limits you are given.