

Problem 8: Trig. equations

(✓✓)

(i) Show that, if $\tan^2 \theta = 2 \tan \theta + 1$, then $\tan 2\theta = -1$.

(ii) Find all solutions of the equation

$$\tan \theta = 2 + \tan 3\theta$$

which satisfy $0 < \theta < 2\pi$, expressing your answers as rational multiples of π .

(iii) Find all solutions of the equation

$$\cot \phi = 2 + \cot 3\phi$$

which satisfy $-\frac{3\pi}{2} < \phi < \frac{\pi}{2}$, expressing your answers as rational multiples of π .

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Comments

There are three distinct parts. It is pretty certain that they are related, but it is not obvious what the relationship is. Part (i) must surely help with part (ii) in some way that will only become apparent once part (ii) is under way.

In the absence of any other good ideas, it looks right to start part (ii) by expressing the double and triple angle tans in terms of single angle tans. You should remember the formula

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

You can use this for $\tan 3\theta$ and hence (later on) for $\cot 3\theta$. If you ever forget the $\tan(A + B)$ formula, you can quickly work it out from the corresponding sin and cos formulae:

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

You are not expected to remember the more complicated triple angle formulae (I certainly don't).

You may well find yourself trying to solve cubic equations at some stage in this question; no need to panic — there is sure to be one easily spottable root, in which case you can reduce the cubic to a quadratic.

Interesting, isn't it, that the range of ϕ for part (iii) is not the obvious $0 < \theta < 2\pi$? Maybe that is significant.

Solution to problem 8

We will write t for $\tan \theta$ (or $\tan \phi$) throughout.

(i) $\tan 2\theta = \frac{2t}{1-t^2} = -1$ (since $2t = t^2 - 1$ by minor rearrangement of the given equation).

(ii) We first work out $\tan 3\theta$. We have

$$\tan 3\theta = \tan(\theta + 2\theta) = \frac{\tan \theta + \tan 2\theta}{1 - \tan \theta \tan 2\theta} = \frac{t + \frac{2t}{1-t^2}}{1 - t \frac{2t}{1-t^2}} = \frac{3t - t^3}{1 - 3t^2},$$

so the equation becomes

$$\frac{3t - t^3}{1 - 3t^2} = t - 2, \quad \text{i.e. } t^3 - 3t^2 + t + 1 = 0.$$

One solution (by inspection) is $t = 1$. Thus one set of roots is given by $\theta = n\pi + \frac{1}{4}\pi$.

There are no other obvious integer roots, but we can reduce the cubic equation to a quadratic equation by dividing out the known factor $(t-1)$. I would start by writing $t^3 - 3t^2 + t + 1 \equiv (t-1)(t^2 + at - 1)$ since the coefficients of t^2 and of t^0 in the quadratic bracket are obvious. Then I would multiply out the brackets to find that $a = -2$. Now we see the connection with part (i): $t^2 + at - 1 = t^2 - 2t - 1 = 0 \Rightarrow \tan 2\theta = -1$, and hence $2\theta = n\pi - \frac{1}{4}\pi$.

The roots are therefore $\theta = n\pi + \frac{1}{4}\pi$ and $\theta = \frac{1}{2}n\pi - \frac{1}{8}\pi$. The multiples of π in the given range are $\{\frac{1}{4}, \frac{3}{8}, \frac{7}{8}, \frac{5}{4}, \frac{11}{8}, \frac{15}{8}\}$.

(iii) For the last part, we could set $\cot \phi = \frac{1}{\tan \phi}$ and $\cot 3\phi = \frac{1}{\tan 3\phi}$ in the given equation, thereby obtaining

$$\frac{1}{t} = 2 + \frac{1 - 3t^2}{3t - t^3}.$$

This simplifies to the cubic equation $t^3 + t^2 - 3t + 1 = 0$. There is an integer root $t = 1$, and the remaining quadratic is $t^2 + 2t - 1 = 0$. Learning from the first part, we write this as $\frac{2t}{1-t^2} = 1$, which means that $\tan 2\phi = n\pi + \frac{1}{4}\pi$. Proceeding as before gives (noting the different range) the following multiples of π : $\{\frac{1}{4}, \frac{1}{8}, -\frac{3}{8}, -\frac{3}{4}, -\frac{7}{8}, -\frac{11}{8}\}$.

Post-mortem

There was a small but worthwhile notational point in this question: it is often possible to use the abbreviation t for $\tan$ (or s for $\sin$, etc), which can save a great deal of writing.

There are two other points worth recalling. First is the way that part (i) fed into part (ii), but had to be mildly adapted for part (iii). This is a typical device used in STEP questions, aimed to see how well you learn new ideas. Second is what to do when faced with a cubic equation. There is a formula for the roots of a cubic, but no one knows it nowadays. Instead, you have to find at least one root by inspection. Having found one root, you have a quick look to see if there are any other obvious roots and, if not, then divide out the known factor to obtain a quadratic equation.

The detectives amongst you will have worked out the reason for the peculiar choice $-\frac{3}{2}\pi < \phi < \frac{1}{2}\pi$ for part (iii). The reciprocal relation between $\tan$ and $\cot$ we used at the start of part (iii) is not the only way to relate these two trigonometric functions. We could have instead used $\cot A = \tan(\frac{1}{2}\pi - A)$.

The equation of part (ii) transforms exactly into the equation of part (iii) if we set $\phi = \frac{1}{2}\pi - \theta$. Furthermore, the given range of ϕ corresponds exactly to the range of θ given in part (ii). We can therefore write down the solutions for part (iii) directly from the solutions for part (ii).