

Problem 5: The modulus function

(✓)

Find all the solutions of the equation

$$|x + 1| - |x| + 3|x - 1| - 2|x - 2| = x + 2 .$$

1997 Paper I

Comments

This looks more difficult than it is.

There is no clever way to deal with the modulus function: you have to look at the different cases individually. For example, for $|x|$ you have to look at $x \leq 0$ and $x \geq 0$. The most straightforward approach would be to solve the equation in each of the different regions determined by the modulus signs: $x \leq -1$, $-1 \leq x \leq 0$, $0 \leq x \leq 1$, etc. You might find a graphical approach helps you to picture what is going on (I didn't).

Solution to problem 5

Let

$$f(x) = |x+1| - |x| + 3|x-1| - 2|x-2| - (x+2).$$

We have to solve $f(x) = 0$ in the five regions of the x -axis determined by the modulus functions, namely

$$-\infty < x \leq -1; \quad -1 \leq x \leq 0; \quad 0 \leq x \leq 1; \quad 1 \leq x \leq 2; \quad 2 \leq x < \infty.$$

In the separate regions, we have

$$f(x) = \begin{cases} -(x+1) + x - 3(x-1) + 2(x-2) - (x+2) & = -2x - 4 & \text{for } -\infty < x \leq -1 \\ (x+1) + x - 3(x-1) + 2(x-2) - (x+2) & = -2 & \text{for } -1 \leq x \leq 0 \\ (x+1) - x - 3(x-1) + 2(x-2) - (x+2) & = -2x - 2 & \text{for } 0 \leq x \leq 1 \\ (x+1) - x + 3(x-1) + 2(x-2) - (x+2) & = 4x - 8 & \text{for } 1 \leq x \leq 2 \\ (x+1) - x + 3(x-1) - 2(x-2) - (x+2) & = 0 & \text{for } 2 \leq x < \infty \end{cases}$$

Solving in each region gives:

1. $-\infty < x \leq -1$: here, $f(x) = 0$ only if $x = -2$. This is a solution since the point $x = -2$ lies in the region $x \leq -1$.
2. $-1 \leq x \leq 0$: here, $f(x) = -2$ ($\neq 0$) so there is no solution.
3. $0 \leq x \leq 1$: here, $f(x) = 0$ only if $x = -1$. This is not a solution since the point $x = -1$ does not lie in the region $0 \leq x \leq 1$.
4. $1 \leq x \leq 2$: here, $f(x) = 0$ only if $x = 2$, which is a solution since $x = 2$ lies in the range $1 \leq x \leq 2$.
5. $2 \leq x < \infty$: here, $f(x) = 0$ identically (i.e. for all values of x), so the equation is satisfied for all x in the region.

Collecting these results together shows that the equation $f(x) = 0$ is satisfied only by $x = -2$ and by any x in the region $x \geq 2$.

Post-mortem

As mentioned before, this should not be found too difficult once you identify the different regions and consider each on its own. Some care is needed, though, and it would be sensible to go back and check that the solutions do indeed satisfy the original equation.

Two small points of technique:

- Since it was necessary to refer to the original equation quite a few times, I found it useful to define a function f so that the equation became $f(x) = 0$.
Alternatively, you could label the equation with a (*), say, when you first write it down and then later say 'Equation (*) becomes ...'.
- It helped me to set out the different cases very clearly. I numbered them later but it might well have saved writing to number them at the beginning.