

Problem 3: Mathematical deduction

(✓)

- (i) Write down the average of the integers $n_1, (n_1 + 1), \dots, (n_2 - 1), n_2$. Show that

$$n_1 + (n_1 + 1) + \dots + (n_2 - 1) + n_2 = \frac{1}{2}(n_2 - n_1 + 1)(n_1 + n_2).$$

- (ii) Write down and prove a general law of which the following are special cases:

$$1 = 0 + 1$$

$$2 + 3 + 4 = 1 + 8$$

$$5 + 6 + 7 + 8 + 9 = 8 + 27$$

$$10 + 11 + \dots + 16 = 27 + 64.$$

Hence prove that

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

1994 Paper I

Comments

You could use induction to prove your general law, but it is not necessary. The obvious alternative involves use of part (i). You might think of induction for the second bit of part (ii), but the question specifically tells you to use the previous result.

There are various ways of summing k th powers of integers, besides the trick given here, which does not readily generalise to powers other than the third. A simple way is to assume that the result is a polynomial of degree $k + 1$. You can then find the coefficients by substituting in $k + 1$ values of n to obtain $k + 1$ simultaneous equations for the coefficients of the polynomial. We can guess that the leading coefficient will in general be $(k + 1)^{-1}$, because the sum approximates the area under the graph $y = x^k$ from $x = 1$ to $x = n$, which is given by the integral of x^k .

Solution to problem 3

(i) The average is $\frac{1}{2}(n_1 + n_2)$. (This is the mid-point of a ruler with n_1 at one end and n_2 at the other.) The sum of any numbers is the average of the numbers times the number of numbers, as given.

(ii) A general rule is

$$\sum_{k=m^2+1}^{(m+1)^2} k = m^3 + (m+1)^3.$$

We can prove this result by applying the formula derived in the first part of the question:

$$\begin{aligned} \sum_{k=m^2+1}^{(m+1)^2} k &= \frac{1}{2}[(m+1)^2 - m^2][(m+1)^2 + (m^2+1)] \\ &= \frac{1}{2}[2m+1][2m^2+2m+2] \\ &= 2m^3 + 3m^2 + 3m + 1 \\ &= m^3 + (m+1)^3. \end{aligned}$$

For the last part, we can obtain sums of the cubes by adding together the general law for consecutive values of m :

$$\begin{aligned} 1 + (2+3+4) + (5+6+7+8+9) + \cdots + ((N-1)^2+1) + \cdots + N^2 \\ = (0+1) + (1+8) + (8+27) + \cdots + ((N-1)^3+N^3). \end{aligned}$$

The left-hand side is just the sum of integers from 1 to N^3 , so using the first part gives

$$\frac{1}{2}N^2(N^2+1) = 2(1^3+2^3+\cdots+N^3) - N^3$$

i.e.

$$\frac{1}{2}N^2(N^2+2N+1) = 2(1^3+2^3+\cdots+N^3)$$

as required.

Post-mortem

To find the general rule requires an understanding of the algebra you have just done: why does it work in the way it does?

Of course, there are many other 'general rules' besides the one given in the solution. The situation is similar to the 'find the next number in the sequence' questions which come up in IQ tests. As no less a figure than the philosopher Wittgenstein has pointed out, there is no correct answer. Given any finite sequence of numbers, a formula can always be found which will fit all the given numbers and which makes the next number (e.g.) 42.

Nevertheless, when the above question was set, almost everyone produced the same general result (or no result at all). Mathematicians seem to know what sort of thing is required. Electronic computers might not have the faintest idea what to do, though they would no doubt complete the algebra rapidly.

Note: This question has been changed slightly from the original version in the 1994 paper to include part (i), and to change the wording slightly in the second part. Having now included a part (i) the hint given in the original question has now been removed.