

Problems

Problem 1: An integer equation

(✓)

- (i) Find all sets of positive integers a, b and c that satisfy the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

- (ii) Determine the sets of positive integers a, b and c that satisfy the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 1.$$

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Comments

Age has not diminished the value of this old chestnut. It requires almost no mathematics in the sense of examination syllabuses, but instead tests a vital asset for a mathematician, namely the capacity for systematic thought. For this reason, it tends to crop up in mathematics contests, where competitors come from different backgrounds. I most recently saw it in the form 'Find all the positive integer solutions of $bc + ca + ab = abc$ '. (You see the connection?)

You will want to make use of the symmetry between a, b and c : if, for example, $a = 2, b = 3$ and $c = 4$, then the same solution can be expressed in five other ways, such as $a = 3, b = 4$ and $c = 2$. You do not want to derive all these separately, so you need to order a, b and c in some way.

The reason the question uses the word 'determine', rather than 'find' for part (ii) is that in part (i) you can write down all the possibilities explicitly, whereas for part (ii) there are, in some cases, an infinite number of possibilities which obviously cannot be explicitly listed, though they can be described and hence determined.

Solution to problem 1

Let us take, without any loss of generality, $a \leq b \leq c$.

(i) How small can a be?

First set $a = 1$. This gives no solutions, because it leaves nothing for $1/b + 1/c$.

Next set $a = 2$ and try values of b (with $b \geq 2$, since we have assumed that $a \leq b \leq c$) in order:

if $b = 2$, then $1/c = 0$, which is no good;

if $b = 3$, then $c = 6$, which works;

if $b = 4$, then $c = 4$, which works;

if $b \geq 5$, then $c \leq b$, so we need not consider this.

Then set $a = 3$, and try values of b (with $b \geq 3$) in order:

if $b = 3$, then $c = 3$, which works;

if $b \geq 4$, then $c \leq b$, so we need not consider this.

Finally, if $a \geq 4$, then at least one of b and c must be $\leq a$, so we need look no further.

The only possibilities are therefore $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$.

(ii) Clearly, we must include all the solutions found in part (i). We proceed systematically, as in part (i).

First set $a = 1$. This time, any values of b and c will do (though we decided to choose $b \leq c$).

Next set $a = 2$ and try values of b (with $b \geq 2$, since we have assumed that $a \leq b \leq c$) in order:

if $b = 2$, then any value of c will do (with $b \leq c$).

if $b = 3$, then 3, 4, 5 and 6 will do for c , but 7 is too big;

if $b = 4$, then $c = 4$ will do, but 5 is too big for c ;

if $b \geq 5$, then $c \leq b$, so we need not consider this.

Then set $a = 3$ and try values of b (with $b \geq 3$) in order:

if $b = 3$, then $c = 3$, which works, but $c = 4$ is too big;

if $b \geq 4$, then $c \leq b$, so we need not consider this.

As before $a = 4$ and $b \geq 4$, $c \geq 4$ gives no possibilities.

Therefore the extra sets for part (ii) are of the form $(1, b, c)$, $(2, 2, c)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$.

Post-mortem

A slightly different approach for part (i), which would also generalise for part (ii), is to start with the case $a = b = c$, for which the only solution is $a = b = c = \frac{1}{3}$. If a , b and c are not all equal then one of them, which we may take to be a , must be greater than $\frac{1}{3}$, i.e. $\frac{1}{2}$. The two remaining possibilities follow easily.

As long as you are methodical, it doesn't matter how you approach the question.

Having done the question, you might well want to investigate whether it generalises: could we replace the 1 on the right hand side with 2? Could we have four reciprocals instead of three? After a few scribbles, I decided that these other cases are not very interesting; but you might find something I missed.