About this book

This book has two aims.

- The general aim is to help bridge the gap between school and university mathematics.

  You might wonder why such a gap exists. The reason is that mathematics is taught at school for various purposes: to improve numeracy; to hone problem-solving skills; as a service for students going on to study subjects that require some mathematical skills (economics, biology, engineering, chemistry — the list is long); and, finally, to provide a foundation for the small number of students who will continue to a specialist mathematics degree. It is a very rare school that can achieve all this, and almost inevitably the course is least successful for its smallest constituency, future mathematics undergraduates.

- The more specific aim is to help you to prepare for STEP or other examinations required for university entrance in mathematics. To find out more about STEP, read the next section.

It used to be said that mathematics and cricket were not spectator sports; this is still true of mathematics. To progress as a mathematician, you have to strengthen your mathematical muscles. It is not enough just to read books or attend lectures. You have to work on problems yourself.

One way of achieving the first of the aims set out above is to work on the second, and that is how this book is structured. It consists almost entirely of problems for you to work on.

The problems are all based on STEP questions. I chose the questions either because they are 'nice'— in the sense that you should get a lot of pleasure from tackling them (I did), or because I felt I had something interesting to say about them.

The first two problems (the 'worked problems') are in a stream of consciousness format. They are intended to give you an idea of how a trained mathematician would think when tackling them. This approach is much too long-winded to sustain for the remainder of the book, but it should help you to see what sort of questions you ought to be asking yourself as you work on the later problems.

Each subsequent problem occupies two pages. On the first page is the STEP question, followed by a comment. The comments may contain hints, they may direct your attention to key points, and they may include more general discussions. On the next page is a solution; you have to turn over, so that your eye cannot accidentally fall on a key line of working. The solutions give enough working for you to be able to read them through and pick up at least the gist of the method; they may not give all the details of the calculations. For each problem, the given solution is of course just one way of producing the required result: there may be many other equally good or better ways. Finally, if there is space on the page after the solution (which is sometimes not the case, especially if diagrams have to be fitted in), there is a postmortem. The postmortems may indicate what aspects of the solution you should be reviewing and they may tell you about the ideas behind the problems.

I hope that you will use the comments and solutions as springboards rather than feather beds. You will only really benefit from this book if you have a good go at each problem before looking at the comment and certainly before looking at the solution. The problems are chosen so that there is something for you to learn from each one, and this will be lost to you for ever if you simply read the solution without thinking about the problem on your own.

I have given each problem a difficulty rating ranging from ✓ to ✓✓✓. Difficulty in mathematics is in the eye of the beholder: you might find a question difficult simply because you overlooked some key step, which on another day you would not have hesitated over. You should not therefore be discouraged if you are stuck on a ✓-question, though you should probably be encouraged if you get through one of the rare ✓✓✓-questions without mishap.

This book is about depth not breadth. I have not tried to teach you any new topics. Instead, I want to lead you towards a deeper understanding of the material you already know. I therefore restricted myself to problems requiring knowledge of the specific and rather limited syllabus that is laid out at the end of this book. This syllabus is for STEP papers that were set before 2019, for 2019 onwards there are new STEP specifications which can be found at www.admissionstesting.org/for-test-takers/step/about-step.

Calculators are not required for any of the problems in this book and calculators are not permitted in STEP examinations. In the early days of STEP, calculators were permitted but they were not required for any question. It was found that candidates who tried to use calculators sometimes ended up missing the point of the question or getting a silly answer. My advice is to remove the battery so that you are not tempted.

I started this section by listing the aims of the book. You may have noticed that teaching you mathematics is not one of them. I can't remember where I heard the following rather nice analogy. In 1464, a huge block of Carrara marble was carefully chosen from a quarry in Tuscany and transported to Florence, where it lay almost untouched for many years. In 1501 it was given to the sculptor Michelangelo. He worked hard on it, chipping away and chipping away for three years, until at last, inside the block, he found a beautiful statue of David. You can see a picture at:
http://www.accademia.org/explore-museum/artworks/michelangelos-david/

And the analogy? I can't teach you mathematics with this book, but I believe that much hard work on your part, chipping away at the problems, will eventually reveal the mathematician that is within you.

I hope you enjoy using this book as much as I have enjoyed putting it together.

STEP

What is STEP?

STEP (Sixth Term Examination Paper) is an examination used by Cambridge University as part of its procedure for admitting students to study mathematics. Applicants are interviewed in December, and may then be offered a place conditional on the results of their public examinations (A-level, International Baccalaureate, etc) and STEP. The examinations are sat in June and offers are confirmed in August when all the examination results are available.

STEP is used for conditional offers not just by Cambridge, but also by Warwick University as part of a range of offers, and to a lesser extent by some other English universities. Many other university mathematics departments recommend that their applicants practise the past papers even if they do not take the examination. In 2019, more than 4500 scripts were marked, compared to about 500 students holding STEP offers from Cambridge.

The first STEPs were taken in 1987, and there were specimen papers before that from which some of the questions in this book are drawn. At that time, there were STEPs in many subjects but by 2001 only the mathematics papers remained. The examination has remained more or less stable over more than 30 years: it has not been blown about by the various fads in the public examination systems that came and went during that time.

I do not want you to think that this book is about examinations; and it is definitely not about how to pass examinations. But since all the questions are taken from STEP papers, I will say a little more about STEP.

There are three STEP papers, 1, 2 and 3, in increasing order of difficulty. More information about STEP, including past papers and up-to-date specifications can be found at www.admissionstesting.org/for-test-takers/step/about-step. The specifications for STEP contain Pure Mathematics, Mechanics and Probability/Statistics.

It has to be said, though, that the statistics questions are very likely to require knowledge of probability rather than statistics (for example, there are very few questions on statistical tests of given data). This is because the underlying theory of statistics is quite difficult, and therefore unsuitable for examination at this level, whereas the application of statistical tests is rather routine and therefore unsuitable for different reasons.

What is the purpose of STEP?

From the point of view of admissions to a university mathematics course, STEP has three purposes.

- It is used as a hurdle for entrance to university mathematics courses, and sometimes for other mathematics-based courses. There is strong evidence that success in STEP correlates very well with university examination results.[1]

[1] Recent studies comparing rank in STEP with rank in first-year Cambridge mathematics examinations reveal a Spearman correlation coefficient of 0.63, which is very high in comparison with other predictors of university examination results.

- It acts as preparation for the university course, because the style of mathematics found in STEP questions is similar to that of undergraduate mathematics.
- It tests motivation. It is important to prepare for STEP (by working through old papers, for example), which can require considerable dedication. Those who are willing to make the effort are more likely to thrive on a difficult university mathematics course.

STEP *vs* A-level

A-level[2] tests mathematical knowledge and technique by asking you to tackle fairly stereotyped problems. STEP asks you to apply the same knowledge and technique to problems that are, ideally, unfamiliar.

Here is an A-level question, in which you follow the instructions in the question:

> By using the substitution $u = 2x - 1$, or otherwise, find
> $$\int \frac{2x}{(2x-1)^2}\,\mathrm{d}x.$$

And here, for comparison, is a STEP question, which requires both competence in basic mathematical techniques and mathematical intuition. Note that help is given for the first integral, so that everyone starts at the same level. Then, for the second integral, candidates have to show that they understand why the substitution used in the first part worked, and how it can be adapted.

> Use the substitution $x = 2 - \cos\theta$ to evaluate the integral
> $$\int_{3/2}^{2} \left(\frac{x-1}{3-x}\right)^{\frac{1}{2}} \mathrm{d}x.$$
> Show that, for $a < b$,
> $$\int_{p}^{q} \left(\frac{x-a}{b-x}\right)^{\frac{1}{2}} \mathrm{d}x = \frac{(b-a)(\pi + 3\sqrt{3} - 6)}{12},$$
> where $p = (3a+b)/4$ and $q = (a+b)/2$.

The differences between STEP and A-level are:

1. STEP questions are much longer.
2. STEP questions are much less routine.
3. STEP questions may require considerable dexterity in performing mathematical manipulations.
4. Individual STEP questions may require knowledge of several different areas of mathematics (especially the mechanics and statistics questions, which will often require advanced pure mathematical techniques).
5. The marks available for each part of the question are not disclosed on the paper.
6. There is a lot of choice on STEP papers (Candidates are expected to attempt 6 questions out of 11 or 12).

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[2] I use the term ‘A-level’ here as a shorthand for a typical school mathematics examination. The particular examinations you take may well be very different in style and format but, even if that is the case, I am sure some of what follows will strike a chord with you.

7. Calculators are not permitted in STEP examinations.

These differences matter, because in mathematics more than in any other subject it is very important to match the difficulty of the question with the ability of the candidates. For example, you could reasonably set the question 'Was Henry VIII a good king?' on a lower-school history paper, an A-level paper, or as a PhD topic. The answers would (or should) differ according to the level. On mathematics examination papers, the question has to be tailored to the expected level in order to discriminate between the candidates: if it is too easy, nearly all candidates will score very high marks; if it is too hard, nearly all candidates will make little progress on any of the questions.

STEP *vs* Olympiad

STEP questions are not like A-level questions, as we have seen, but they are not like Olympiad questions either. Typically, an Olympiad question comes with no 'scaffolding' (as it is nowadays called) at all. There are no little hints or toe-holds to allow you to find a way into the problem. A typical STEP question (see the next section) does come with a bit of scaffolding; it is more of a learning process. You could say that it is more about the journey than about the arrival.

STEP Questions

STEP questions do not fall into any one category. Typically, there will be a range of types on each of the papers. Here are some thoughts, in no particular order.

- My favourite sort of question is in two (or maybe more) parts: in the first part, you are asked to perform some unfamiliar task and are told how to do it (integration using a given substitution, or expressing a quartic as the algebraic sum of two squares, for example); for the later parts, you are expected to demonstrate that you have understood and learned from the first part by applying the method to a new and perhaps more complicated task.
- Another favourite of mine is the question which has different answers according to the value of a certain number (or *parameter*). A common example involves sketching a graph whose shape depends on whether a parameter is positive or negative. Ideally, the different values of the parameter are not given in the question, and you have to identify them for yourself.
- Another good type of question requires you to do some preliminary special-case work and then prove a general result.
- In another type, you have to show that you can understand and use new notation or a new theorem.
- Questions with several unrelated parts (for example, three integrals using different techniques) are generally avoided; but if they occur, there tends to be a 'sting in the tail' involving putting all the parts together in some way.
- Some questions do not rely on any part of the syllabus: instead they might require 'common sense', involving counting or seeing patterns, or they might involve some aspect of more elementary mathematics with an unusual slant. Such questions try to test capacity for clear and logical thought without using much mathematical knowledge (like the calendar question mentioned in the section on preparation below, or questions concerning islands populated by toads 'who always tell the truth' and frogs 'who always fib').

- Some questions are devised to check that you do not simply apply routine methods blindly. For example, a function might have a maximum value at the end of the interval upon which it is defined, even though its derivative might be non-zero there. Finding the maximum in such a case is not simply a question of routine differentiation.

- There are always questions specifically on integration or differentiation, and many others (including mechanics and probability) that use calculus as a means to an end.

- Graph-sketching is regarded by mathematicians as a fundamental skill and there are nearly always questions that require a sketch.

- Basic ideas from analysis, such as

$$0 \leqslant \mathrm{f}(x) \leqslant k \Rightarrow 0 \leqslant \int_a^b \mathrm{f}(x)\mathrm{d}x \leqslant (b-a)k$$

  or

$$\mathrm{f}'(x) > 0 \Rightarrow \mathrm{f}(b) > \mathrm{f}(a) \text{ for } b > a$$

  or the relationship between an integral and a sum often come up, though knowledge of such results is never assumed — candidates may be told that they can use the result without proof, or a sketch 'proof' may be requested.

- As mentioned above, questions on statistical tests are rare, because questions that require real understanding (rather than 'cookbook' methods) tend to be too difficult. More often, the questions in the Probability and Statistics section are about probability.

- The mechanics questions normally require a firm understanding of the basic principles (when to apply conservation of momentum and energy, for example) and may well involve a differential equation. Projectile questions are often set, but are never routine.

Advice to candidates

First appearances

I am often asked whether STEP is 'difficult'. Of course, it depends on what is meant by 'difficult'; it is not difficult compared with the mathematics I do every day. But to be on the safe side, I always answer 'yes' before explaining further.

Your first impression on looking at a STEP paper is likely to be that it does indeed look very difficult. Don't be discouraged! Its difficult appearance is largely due to it being very different in style from what you are used to.

At the time of writing, a typical A-level examination lasts between 90 mins and two hours and contains between 7 and 13 questions. That is about 10 minutes per question. If you are considering studying mathematics at a top university, it is likely that you will manage to do them all and get them nearly all right in the time available. A STEP examination lasts 3 hours, and you are only supposed to do six questions. This means that each question is designed to take about 30 minutes. If you compare a 10-minute question with a 30-minute question, the 30-minute question will look substantially harder.

When looking through past papers you may be put off by the number of different topics covered on the paper. Before the 2017-18 A-level reforms there was a considerable amount of variation in the different specifications of each examination board, and STEP was designed to provide sufficient questions for all of those specifications. From the summer of 2019 onwards there is a lot less variation (for the pure topics at least). Also, you get to chose which questions to do so you do not have to be overly concerned about the inclusion of some topics you are less fond of.

Once you get used to the idea that STEP is very different from A-level, it becomes much less daunting.

Preparation

The best preparation for STEP (apart, of course, from working through this excellent book) is to work slowly through past papers.[3] Hints and answers are available for the more recent years, but you should use these with discretion: doing a question with hints and answers in front of you is nothing like doing it yourself, and you may well miss the whole point of the question (which is to make you think about mathematics). In general, thinking about the problem is much more important than getting the answer.

It is worth emphasising that there is no 'hidden agenda': a candidate who does two complete probability questions and two complete mechanics questions will obtain the same mark and grade as one who does four complete pure questions. Similarly a candidate who scores 15 on each of 4 questions will get the same grade as one who scores 10 on each of 6 questions.

Just as the examiners have no hidden agenda concerning question choice, so they have no hidden agenda concerning your method of answering the question. If you can get to the end of a question correctly you will get full marks whatever method you use.[4] Some years ago one of the questions asked candidates to find the day of the week of a given date (say, the 5th of June 1905). A candidate who simply counted backwards day by day from the date of the exam would have received full marks for that question (but would not have had time to do any other questions).

You may be worried that the examiners expect some mysterious thing called rigour. Do not worry: STEP is an exam for schools, not universities, and the examiners understand the difference. Nevertheless, it is extremely important that you present ideas clearly, and show working at all stages.

Presentation

You should set out your answer legibly and logically (don't scribble down the first thought that comes into your head)—this not only helps you to avoid silly mistakes but also signals to the examiner that you know what you are doing (which can be effective even if you haven't the foggiest idea what you are doing). It helps if you can read your own writing; be sure that you can distinguish between '2' and 'z' and '5' and 's'.

Examiners are not as concerned with neatness as you might fear. However, if you receive complaints from your teachers that your answers are difficult to follow then you should listen.[5] Remember that more space usually means greater legibility. Try writing on alternate lines (this leaves a blank line for corrections).

Try to read your answers with a hostile eye. Have you made it clear when you have come to the end of

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[3] These and the other publications mentioned below are obtainable from the Cambridge Assessment Admissions Testing website www.admissionstesting.org/for-test-takers/step/about-step. You may also find the STEP Support Programme (maths.org/step) useful. This has been developed by the University of Cambridge to help students prepare for STEP.

[4] Though you must obey instructions in the question: for example, if it says 'Hence prove ...', then you must use the previous result in your proof. One example is 2018 STEP 3 question 5 which asked candidates to deduce the Arithmetic Mean-Geometric Mean inequality. Those candidates who did not 'deduce' this from previous work did not do as well.

[5] Begin rant: I am very surprised at the scrappy and illegible work that I receive from a few of my students. It seems so disrespectful to expect me to spend ages trying to decipher their work when they could have spent a little more time making it presentable, for example by copying it out neatly or writing more slowly. Why is my time less important than theirs? End rant.

a particular argument? Try underlining your conclusions. Have you explained what you are trying to do? For example, if a question asks 'Is A true?' try beginning your answer by writing 'A is true'—if you think that it is true—so that the examiner knows which way your argument leads. If you used an idea (for example, integration by substitution), did you tell the examiner that this is what you were doing?

You may find that you stop a question and then restart it later—in which case a note to the examiner 'continued later in booklet' will make your solution easier to follow. Also, if you decide to cross out some working do it neatly (for example by putting one diagonal line through it) as then you have the option of writing 'please mark this' if you change your mind.

What to do if you cannot get started on a problem

Try the following, in order:

- Reread the question to check that you understand what is wanted.
- Reread the question to look for clues – the way it is phrased, or the way a formula is written, or other relevant aspects of the question. (You may think that the setters are trying to set difficult questions or to catch you out. Usually, nothing could be further from the truth: they are probably doing all in their power to make it easy for you by trying to tell you what to do).
- Try to work out exactly what it is that you don't understand.
- Simplify the notation – for example, by writing out sums explicitly.
- Look at special cases (choose special values which simplify the problem) in order to try to understand why a result is true.
- Write down your thoughts – in particular, try to express the exact reason why you are stuck.
- Go on to another question and go back later.
- If you are preparing for the examination (but not in the actual examination!) take a short break.[6]
- Discuss it with a friend or teacher (again, better not do this in the actual examination) or consult the hints and answers, but make sure you still think it through yourself.

  BUT REMEMBER: following someone else's solution is not remotely the same thing as doing the problem yourself. Once you have seen someone else's solution to a problem, then you are deprived, for ever, of much of the benefit that could have come from working it out yourself.

  Even if, ultimately, you get stuck on a particular problem, you derive vastly more benefit from seeing a solution to something with which you have already struggled, than by simply following a solution to something to which you've given very little thought.

What if a problem isn't coming out?

If you have got started but the answer doesn't seem to be coming out, then try the following:

[6] J. E. Littlewood (1885–1977), distinguished Cambridge mathematician and author of the highly entertaining *A Mathematician's Miscellany* (Cambridge: Cambridge University Press, 1986) used to work seven days a week until an experiment revealed that when he took Sundays off, the good ideas had a way of coming on Mondays.

- Check your algebra. In particular, make sure that what you have written works in special cases. For example: if you have written the series for $\log(1+x)$ as

$$1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + \cdots$$

then a quick check will reveal that it doesn't work for $x = 0$; clearly, the 1 should not be there.[7]

A note on the subject of algebra. In many of the problems in this book, the algebra is quite stiff: you have to go through many lines of calculation before you get to an expression recognisably close to your target. Really, the only way to manage this efficiently is to check each line carefully before going on the next line. Otherwise, you can waste hours.[8]

- Make sure that what you have written makes sense. For example, in a problem which is dimensionally consistent, you cannot add x (with dimension length, say) to x^2 or to $\exp x$ (which itself does not make sense — the argument of exp has to be dimensionless). Even if there are no dimensions in the problem, it is often possible to mentally assign dimensions and hence enable a quick check.

Be wary of applying familiar processes to unfamiliar objects (very easy to do when you are feeling at sea): for example, it is all too easy, if you are not sure where your solution is going, to solve the vector equation $\mathbf{a}.\mathbf{x} = 1$ by dividing both sides by vector $\mathbf{a}$; a bad idea.

- Analyse exactly what you are being asked to do. Try to understand the hints, explicit and implicit. Remember to distinguish between terms such as explain/prove/define/etc. (There is essentially no difference between 'prove' and 'show': the former tends to be used in more formal situations, but if you are asked to 'show' something, a proper proof is required.)

- Remember that different parts of a question are often linked. There may be guidance in the notation and choice of names of variables in the question.

- If you get irretrievably stuck in the exam, state in words what you are trying to do and move on (at A-level, you don't get credit for merely stating intentions, but STEP examiners are generally grateful for any sign of intelligent life).

What to do after completing a question

It is a natural instinct to consider that you have finished with a question once you have got to an answer. However this instinct should be resisted both from a general mathematical point of view and from the much narrower view of preparing for an examination. Instead, when you have completed a question you should stop for a few minutes and think about it. Here is a check list for you to run through:

- Look back over what you have done, checking that the arguments are correct and making sure that they work for any special cases you can think of. It is surprising how often a chain of completely spurious arguments and gross algebraic blunders leads to the given answer.

- Check that your answer is reasonable. For example if the answer is a probability p then you should check that $0 \leqslant p \leqslant 1$. If your answer depends on an integer n, does it behave as it should when $n \to \infty$? Is it dimensionally correct?

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7 Another check will reveal that for very small positive x, log is positive (since its argument is bigger than 1) whereas the series is negative (once the 1 has been removed), so there is clearly something else wrong.

8 Which is what I normally do; but I am still hoping to heed my own advice in the not-too-distant future.

If, in the exam, you find that your answer is not reasonable, but you don't have time to do anything about it, then write a brief phrase showing that you understand that your answer is unreasonable (for example 'This is wrong because mass must be positive'). It may be that your time will be more profitably spent doing another question than hunting for a sign error.

- Check that you have used all the information given. In many ways the most artificial aspect of examination questions is that you are given exactly the amount of information required to answer the question. If your answer does not use everything you are given then either it is wrong or you are faced with a very unusual examination question.
- Check that you have understood the point of the question. It is, of course, the case that not all exam questions have a point, but many do. What idea did the examiners want you to have? Which techniques did they want you to demonstrate? Is the result of the question interesting in some way? Does it generalise? If you can see the point of the question would your working show the point to someone who did not know it in advance?
- Make sure that you are not unthinkingly applying mathematical tools which you do not fully understand.[9]
- As preparation for the examination, make sure that you actually understand not only what you have done, but also why you have done it that way rather than some other way. This is particularly important if you have had to use a hint or solution.
- If the question has associated hints, answers or examiner's reports then it is a good idea to read these to see what the common approaches and pitfalls were.
- In the examination, check that you have given the detail required. There often comes a point in a question where, if we could show that A implies B, then the result follows. If, after a lot of thought, you suddenly see that A does indeed imply B the natural thing to do is to write triumphantly 'But A implies B so the result follows'[10]. Unfortunately, unscrupulous individuals (not you, of course) who have no idea why A should imply B (apart from the fact that it would complete the question) could, and do, write the exactly the same thing. Go back through the major points of the question making sure that you have written enough for the examiner to be able to follow your reasoning.

[9] Mathematicians should feel as insulted as engineers by the following joke.

A mathematician, a physicist and an engineer enter a mathematics contest, the first task of which is to prove that all odd numbers are prime. The mathematician has an elegant argument: '1's a prime, 3's a prime, 5's a prime, 7's a prime. *Therefore*, by mathematical induction, all odd numbers are prime. It's the physicist's turn: '1's a prime, 3's a prime, 5's a prime, 7's a prime, 11's a prime, 13's a prime, so, to within experimental error, all odd numbers are prime.' The most straightforward proof is provided by the engineer: '1's a prime, 3's a prime, 5's a prime, 7's a prime, 9's a prime, 11's a prime . . . '.

[10] There is an old anecdote about the distinguished Professor X. In the middle of a lecture she writes 'It is obvious that A', suddenly falls silent and after a few minutes she rubs it out and walks out of the room. The awed students hear her pacing up and down outside. Then after twenty minutes she returns and writes 'It is obvious that A' and continues the lecture.

Worked Problems

Worked problem 1 (✓)

Let

$$\mathrm{f}(x) = ax - \frac{x^3}{1+x^2}\,,$$

where a is a constant.
Show that, if $a \geqslant 9/8$, then

$$\mathrm{f}'(x) \geqslant 0$$

for all x.

2000 Paper I

First thoughts

When I play tennis and I see a ball that I think I can hit, I rush up to it and smack it into the net. This is a tendency I try to overcome when I am doing mathematics. In this question, for example, even though I'm pretty sure that I can find $\mathrm{f}'(x)$, I'm going to pause for a moment before I do so. I am going to use the pause to think about two things.

- I'm going to think about what to do when I have found $\mathrm{f}'(x)$.
- I'm going to try to decide what the question is really about.

Of course, I may not be able to decide what to do with $\mathrm{f}'(x)$ until I actually see what it looks like, and I may not be able to see what the question is really about until I have finished it—maybe not even then—some questions are not really about anything in particular. I am also going to use the pause to think a bit about the best way of performing the differentiation. Should I simplify first? Should I make some sort of substitution? Clearly, knowing how to tackle the rest of the question might guide me in deciding the best way to do the differentiation. Or it may turn out that the differentiation is fairly straightforward, so that it doesn't matter how I do it.

Two more points occur to me as I re-read the question:

- I notice that the inequalities are not strict (they are $\geqslant$ rather than $>$). Am I going to have to worry about the difference?
- I also notice that there is an 'If ... then', and I wonder if this is going to cause me trouble. I will need to be careful to get the implication the right way round. I mustn't try to prove that if $\mathrm{f}'(x) \geqslant 0$ then $a \geqslant 9/8$.

  It will be interesting to see why the implication is only one way—why it is not an 'if and only if' question. It may be just 'if' because 'only if' isn't true; or it may be just 'if' because the examiners thought that the question was long enough without the 'only if'.

Don't turn over until you have spent a little time thinking along these lines.

Doing the question

Looking ahead, it is clear that the real hurdle is going to be showing that $f'(x) \geqslant 0$. How am I going to do that? Two ways suggest themselves. First, if $f'(x)$ turns out to be a quadratic function, or an obviously positive multiple of a quadratic function, I should be able to use some standard method: looking at the discriminant ('$b^2 - 4ac$'), etc; or, better, completing the square. But if $f'(x)$ is not of this form, I will have to think of something else: maybe I will have to sketch a graph.

I've just noticed that f is an odd function, i.e. $f(x) = -f(-x)$. (Did you notice that?) That is helpful, because it means that $f'(x)$ is an even function.[11] If it had been odd, there could have been a difficulty, because all odd functions (at least, those with no vertical asymptotes) cross the horizontal axis $y = 0$ at least once and cannot therefore be positive for all x.

Now, how should I do the differentiation? I could divide out the fraction, giving

$$\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}$$

and

$$f(x) = (a-1)x + \frac{x}{1+x^2} = bx + \frac{x}{1+x^2}\,, \tag{$\dagger$}$$

where I have set $b = a - 1$ to save writing. (Is this right? I'll just check that it works when $x = 2$. Yes, it does: $f(2) = 2a - \frac{8}{5} = 2(a-1) + \frac{2}{5}$.) This might save a bit of writing, but I don't at the moment see it helping me towards a positive function. On balance, I think I'll stick with the original form.

Another thought: should I differentiate the fraction using the quotient formula or should I write it as $x^3(1+x^2)^{-1}$ and use the product rule? I doubt if there is much in it. I never normally use the quotient rule—it's just extra baggage to carry around. But on this occasion, since the final form I am looking for is a single fraction and the denominator using the quotient rule is a square and therefore non-negative, I will.

One more thought: since I am trying to obtain an inequality, I must be careful throughout not to cancel any quantity which might be negative; or at least if I do cancel a negative quantity, I must remember to reverse the inequality.

Here goes (at last):

$$\begin{aligned} f'(x) &= a - \frac{3x^2(1+x^2) - 2x(x^3)}{(1+x^2)^2} \\ &= \frac{a + (2a-3)x^2 + (a-1)x^4}{(1+x^2)^2}\,. \end{aligned}$$

I had to do a bit of algebra to obtain the second equation.

This is working out as I had hoped: the denominator is certainly positive and the numerator is a quadratic function of x^2. I can finish this off most elegantly by completing the square. There are two ways of doing this. Just to be clear in my mind, I'm going to write the numerator as

$$A + Bx^2 + Cx^4\,,$$

where $A = a$, $B = 2a - 3$ and $C = a - 1$. Then I can complete the square in two ways:

$$A\big(1 + (B/2A)x^2)\big)^2 + \big(C - B^2/4A\big)x^4$$

or

$$C\big(x^2 + B/2C)\big)^2 + \big(A - B^2/4C\big)\,. \tag{$\ddagger$}$$

[11] You can see this easily by sketching a 'typical' odd function, or by differentiating the Maclaurin expansion of an odd function.

It doesn't seem to matter which I use. The second expression (‡) looks a bit simpler.

If $a \geqslant \frac{9}{8}$, then certainly $C > 0$. That means that the first of the two terms of (‡) is positive: $C(x^2 + B/2C)^2 > 0$.

The second term of (‡) can be written in terms of a as follows:

$$\begin{aligned} A - B^2/4C &= a - \frac{(2a-3)^2}{4(a-1)} \\ &= \frac{4a(a-1) - (2a-3)^2}{4(a-1)} \\ &= \frac{8a-9}{4(a-1)}\,. \end{aligned} \tag{$*$}$$

It is very pleasing to see the numerator $8a - 9$; it is exactly what I want, because it is non-negative if $a \geqslant \frac{9}{8}$. That is what I needed to complete the question.

Post-mortem

Now I can look back and analyse what I have done.

On the technical side, it seems I was right not to use the simplified form of $x^3/(1 + x^2)$ given in (†). This would have lead to the quadratic $(b + 1) + (2b - 1)x^2 + bx^4$, which isn't any easier to handle than the quadratic involving a and just gives an extra opportunity to make an algebraic error.

Although introducing new variables A, B and C seemed at first to complicate the problem, it was useful to have them when it came to completing the square, which would have been a bit of a mess had I worked directly from the quartic expression with a in it.

Actually, I see on re-reading my solution that I have made a bit of a meal out of the ending. I needn't have completed the square at all; I could have used the inequality $a \geqslant 9/8$ immediately after finding $\mathrm{f}'(x)$, since a appears in $\mathrm{f}'(x)$ with a plus sign always:

$$\mathrm{f}'(x) = \frac{a + (2a-3)x^2 + (a-1)x^4}{(1+x^2)^2} \geqslant \frac{\frac{9}{8} + (\frac{9}{4} - 3)x^2 + (\frac{9}{8} - 1)x^4}{(1+x^2)^2} = \frac{(x^2-3)^2}{8(1+x^2)^2} \geqslant 0$$

as required.

However, my unnecessarily elaborate proof, involving completing the square, makes the role of a a bit clearer than it is in the shorter alternative. In fact, I can see how to answer my question about 'if and only if', namely, is it the case that $\mathrm{f}'(x) \geqslant 0$ for all x if *and only if* $a \geqslant \frac{9}{8}$? The answer is: Yes. Looking at (‡), I see that if $\mathrm{f}'(x) \geqslant 0$ for all x, then certainly $C \geqslant 0$, otherwise for large enough x the first (squared) term would be dominant and negative. But if $0 \leqslant C < \frac{1}{8}$, then $B < 0$ and there is a value of x^2 for which the squared term in (‡) vanishes, leaving only the second term which is negative, as can be seen from $(*)$. That means $\mathrm{f}'(x) < 0$ for this value of x, contradicting our assumption. Therefore, $\mathrm{f}'(x) \geqslant 0$ for all x implies that $C \geqslant \frac{1}{8}$ and $a \geqslant \frac{9}{8}$.

I wonder why the examiner wanted me to investigate the sign of $\mathrm{f}'(x)$. The obvious reason is to see what the graph looks like. We can now see what this question is about. It is clear that the examiners really wanted to set the question: 'Sketch the graphs of the function $ax - x^3/(1 + x^2)$ in the different cases that arise according to the value of a' but it was thought too long or difficult. It is worth looking back over my working to see what can be said about the shape of the graph of $\mathrm{f}(x)$ when $a < 9/8$.

(I leave that to you to think about!)

Worked problem 2 (✓)

The n positive numbers $x_1, x_2, \dots, x_n$, where $n \geqslant 3$, satisfy

$$x_1 = 1 + \frac{1}{x_2}, \quad x_2 = 1 + \frac{1}{x_3}, \quad \dots, \quad x_{n-1} = 1 + \frac{1}{x_n},$$

and also

$$x_n = 1 + \frac{1}{x_1}.$$

Show that

(i) $x_1, x_2, \dots, x_n > 1$,

(ii) $x_1 - x_2 = -\dfrac{x_2 - x_3}{x_2 x_3}$,

(iii) $x_1 = x_2 = \cdots = x_n$.

Hence find the value of x_1.

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First thoughts

My first thought is that this question has an unknown number of variables: $x_1, \dots, x_n$. That makes it seem rather complicated. I might, if necessary, try to understand the result by choosing an easy value for n (maybe $n = 3$). If I manage to prove some of the results in this special case, I will certainly go back to the general case: doing the special case might help me tackle the general case, but I don't expect to get many marks in an exam if I just prove the result in one special case.

Next, I see that the question has three sub-parts, then a final one. The final one begins 'Hence ...'. This means that I must use at least one of the previous parts in my working for the final part. It is not clear from the structure of the question whether the three sub-parts are independent; the proof of (ii) and (iii) may require the previous result(s), or it may not.

Actually, I think I can see how to do the very last part. If I assume that (iii) holds, so that $x_1 = x_2 = \cdots = x_n = x$ (say), then each of the equations given in the question is identical and each gives a simple equation for x.

It is surprising, isn't it, that the final result doesn't depend on the value of n? That makes the idea of choosing $n = 3$, just to see what is going on, quite attractive, but I'm not going to resort to that idea unless I get very stuck.

The notation in part (i) is a bit odd. I'm not sure that I have seen anything like it before.[12] But it can only mean that each of the variables $x_1, x_2, \dots x_n$ is greater than 1.

In fact, now I think about it, I am puzzled about part (i). How can an *inequality* help to derive the *equalities* in the later parts? I can think of a couple of ways in which the result $x_i > 1$ could be used. One is that I may need to cancel, say, x_1 from both sides of an equation in which case I would need to know that $x_1 \neq 0$. But looking back at the question, I see I already know that $x_1 > 0$ (it is given right at the

[12] It is just the sort of thing that is used in university texts; but I'm not sure that it would be used in STEP papers nowadays.

start of the question), so this cannot be the right answer. Maybe I need to cancel some other factor, such as $(x_1 - 1)$. Another possibility is that I get two or more solutions by putting $x_1 = x_2 = \cdots = x_n = x$ and I need the one with $x > 1$. This may be the answer: looking back at the question again, I see that it asks for *the* value of x_1 — so I am looking for a single value. I'm still puzzled, but I will remember to keep a sharp look out for ways of using part (i).

One more thing strikes me about the question. The equations satisfied by the x_i are given on two lines ('and also'). This could be for typographic reasons (the equations would not all fit on one line), but more likely it is to make sure that I have noticed that the last equation is a bit different: all the other equations relate x_i to x_{i+1}, whereas the last equation relates x_n to x_1. It goes back to the beginning, completing the cycle. I'm pleased that I thought of this, because this circularity must be important.

Doing the question

I think I will do the very last part first, and see what happens.

Suppose, assuming the result of part (iii), that $x_1 = x_2 = \cdots = x_n = x$. Then substituting into any of the equations given in the question gives

$$x = 1 + \frac{1}{x}$$

i.e. $x^2 - x - 1 = 0$. Using the quadratic formula gives

$$x = \frac{1 \pm \sqrt{5}}{2}\,,$$

which does indeed give two answers (despite the fact that the question asks for just one). However, I see that one is negative and can therefore be eliminated by the condition $x_i > 0$ which was given in the question (not, I note, the condition $x_1 > 1$ from part (i); I still have to find a use for this).

I needed only part (iii) to find x_1, so I expect that either I need both (i) and (ii) directly to prove (iii), or I need (i) to prove (ii), and (ii) to prove (iii).

Now that I have remembered that $x_i > 0$ for each i, I see that part (i) is obvious. Since $x_2 > 0$ then $1/x_2 > 0$ and the first equation given in the question, $x_1 = 1 + 1/x_2$, shows immediately that $x_1 > 1$ and the same applies to x_2, x_3, etc.

Now what about part (ii)? The given equation involves x_1, x_2 and x_3, so clearly I must use the first two equations given in the question:

$$x_1 = 1 + \frac{1}{x_2}\,, \quad x_2 = 1 + \frac{1}{x_3}\,.$$

Since I want $x_1 - x_2$, I will see what happens if I subtract the two equations:

$$x_1 - x_2 = (1 + \frac{1}{x_2}) - (1 + \frac{1}{x_3}) = \frac{1}{x_2} - \frac{1}{x_3} i = \frac{x_3 - x_2}{x_2 x_3} = -\frac{x_2 - x_3}{x_2 x_3}\,. \qquad (*)$$

That seems to work!

One idea that I haven't used so far is what I earlier called the circularity of the equations: the way that x_n links back to x_1. I'll see what happens if I extend the above result. Since there is nothing special about x_1 and x_2, the same result must hold if I add 1 to each of the suffices:

$$x_2 - x_3 = \frac{x_3 - x_4}{x_3 x_4}\,.$$

I see that I can combine this with the previous result:

$$x_1 - x_2 = -\frac{x_2 - x_3}{x_2 x_3} = \frac{x_3 - x_4}{x_2 x_3^2 x_4}\,.$$

I now see where this is going. The above step can be repeated to give

$$x_1 - x_2 = \frac{x_3 - x_4}{x_2 x_3^2 x_4} = -\frac{x_4 - x_5}{x_2 x_3^2 x_4^2 x_5} = \cdots$$

and eventually I will get back to $x_1 - x_2$:

$$x_1 - x_2 = -\frac{x_4 - x_5}{x_2 x_3^2 x_4^2 x_5} = \cdots = \pm\frac{x_1 - x_2}{x_2 x_3^2 x_4^2 x_5^2 \ldots x_n^2 x_1^2 x_2}$$

i.e.

$$(x_1 - x_2)\left(1 \mp \frac{1}{x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 \ldots x_n^2}\right) = 0\ .$$

I have put in a $\pm$ because each step introduces a minus sign and I'm not sure yet whether the final sign should be $(-1)^n$ or $(-1)^{n-1}$. I can check this later (for example, by working out one simple case such as $n = 3$); but I may not need to.

I deduce from this last equation that either

$$x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 \ldots x_n^2 = \pm 1$$

or

$$x_1 = x_2$$

(which is what I want). At last I see where to use part (i): I know that

$$x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 \ldots x_n^2 \neq \pm 1$$

because $x_1 > 1$, $x_2 > 1$, etc. Thus the only possibility is $x_1 = x_2$. Since there was nothing special about x_1 and x_2, I deduce further that $x_2 = x_3$, and so on, as required.

Post-mortem

There were a number of useful points in this question:

1. The first point concerns using the information given in the question. The process of teasing information from what is given is fundamental to the whole of mathematics. It is very important to study what is given (especially seemingly unimportant conditions, such as $x_i > 0$) to see why it has been given. If you find you reach the end of a question without apparently using some given information, then you should look back over your work: it is very unlikely that a condition has been given that is not used in some way. It may not be a necessary condition — and we will see that the condition $x_i > 0$ is not, in a sense, necessary in this question — but it should be sufficient. The other piece of information in the question which you might easily have overlooked is the use of the singular rather than the plural in referring to the solution (' ... find *the value* of ...'), implying that there is just one value, despite the fact that the final equation is quadratic.

2. The second point concerns the structure of the question. Here, the position of the word 'Hence' suggested strongly that none of the separate parts were stand-alone results; each had to be used for a later proof. Understanding this point made the question much easier, because I was always on the look out for an opportunity to use the earlier parts. Of course, in some problems (without that 'hence') some parts may be stand-alone; though this is rare in STEP questions.

   You may think that this is like playing a game according to hidden (STEP) rules, but that is not the case. Precision writing and precision reading is vital in mathematics and in many professions (law, for example). Mathematicians have to be good at it, which is the reason why so many employers want to recruit people with mathematical training.

3. The third point was the rather inconclusive speculation about the way inequalities might help to derive an equality. It turned out that what was actually required was $x_1x_2 \dots x_n \neq \pm 1$. I was a bit puzzled by this possibility in my first thoughts, because it seemed that the result ought to hold under conditions different from those given; for example, $x_i < 0$ for all i (does this condition work??). Come to think of it, why are conditions given on all x_i when they are all related by the given equations? This makes me think that there ought to be a better way of proving the result which would reveal exactly the conditions under which it holds.

4. Then there was the idea (which I didn't actually use) that I might try to prove the result for, say, $n = 3$ to help me understand what was going on. This would not have counted as a proof of the result (or anything like it), but it might have given me ideas for tackling the question.

5. A key observation was that the equations given in the question are 'circular'. It was clear that the circularity was essential to the question and it turned out to be the key to the most difficult part. Having identified it early on, I was ready to use it when the opportunity arose.

6. Finally, I was pleased that I read the whole question carefully before plunging in. This allowed me to see that I could easily do the last part before the preceding parts, which I found very helpful in getting into the question.

Final thoughts

It occurs to me only now, after my post-mortem, that there is another way of obtaining the final result.

Suppose I start with the idea of circularity (as indeed I might have, had I not been otherwise directed by the question) and use the given equations to find x_1 in terms of first x_2, then x_3, then x_4 and eventually in terms of x_1 itself. That should give me an equation I can solve, and I should be able to find out what conditions are needed on the x_i. Try it. You may need to guess a formula for x_1 in terms of x_i from a few special cases, then prove it by induction.[13] You will find it useful to define a sequence of numbers F_i such that $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$. (These numbers are called *Fibonacci numbers*.[14])

You should find that if $x_n = x_1$ for any n (greater than 1), then $x_n = \frac{1 \pm \sqrt{5}}{2}$.

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[13] I went as far as x_7 to be sure of the pattern: I found that $x_7 = \frac{8x_1+5}{5x_1+3}$.

[14] Fibonacci (short for *filius* Bonacci — son of Bonacci) was called the greatest European mathematician of the Middle Ages. He was born in Pisa (Italy) in about 1175 AD. He introduced the series of numbers named after him in his book of 1202 called *Liber Abbaci (Book of the Abacus)*. It was the solution to the problem of the number of pairs of rabbits produced by an initial pair: *A pair of rabbits are put in a field and, if rabbits take a month to become mature and then produce a new pair every month after that, how many pairs will there be in twelve months time?*